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Hamiltonian / Gauge theoretical Gromov-Witten invariants of toric varieties *

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Abstract. We present a purely algebraic approach to the Hamiltonian / Gauge theoretical invariants associated to torus actions on affine spaces. Secondly, we address the issue of computing the invariants: a localization and a genus recursion formula are deduced.

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1. Introduction

This article was originally intended to be an application to the case of toric varieties of the ideas developed in my paper [14], where I have studied the problem of computing the Gromov-Witten invariants of quotient varieties. It has been already observed in *loc. cit.* that this question ties in with certain Hamiltonian Gromov-Witten invariants, a notion which was emerging that time. During the last two years the Hamiltonian/gauge theoretical Gromov-Witten invariants took a much more precise shape due to the articles [4,5,18,22,23]. Their common feature is that of introducing these new invariants by using infinite dimensional methods. However, we have observed in [14] that, at least in the abelian case, the infinite dimensional approach to the problem can be simplified and presented within an algebro-geometric frame. The goal of this article is to present the Hamiltonian Gromov-Witten invariants for smooth and projective toric varieties using only algebraic techniques. This finite dimensional approach has also its own advantage since it allows us also to push the study of these invariants further: namely we show that they obey recursive relations, analogous to those fulfilled by the usual Gromov-Witten invariants.

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In contrast to the usual Gromov-Witten invariants which require the full stable map compactification, the Hamiltonian/gauge theoretical invariants make use of a more amenable compactification which is birational to an irreducible component of the stable map compactification à la Kontsevich-Manin. This feature makes these invariants more suited for computational purposes. The toroidal compactification appearing in the genus zero case was first used by Batyrev in [2] for defining an explicit quantum multiplication on the cohomology ring of toric varieties. It was checked later in [24] that the genus zero invariants constructed in this way, do not coincide in general with the usual Gromov-Witten invariants. The same compactification has appeared in the physics literature in the context of supersymmetry in the articles [25, 17], where the authors were led to the study of the so-called 'gauged linear sigma models' with target a toric variety.

The results of this article can be summarized as follows: let X be a smooth and projective toric variety, and let $A^1(X)$ be its Chow group of divisors. If $T := \mathsf{Hom}_{\mathbb{Z}}\big(A^1(X);\mathbb{C}^*\big)$, then it is well-known that $X = \Omega/T$ for a suitable open subset $\Omega \subset \mathbb{C}^r$ in an affine space. The T-equivariant Chow ring of a point can be presented as $\mathbb{Z}[\chi_1,\dots,\chi_r]/\langle\mathrm{LR}\rangle$, where $\langle\mathrm{LR}\rangle$ is the ideal of so-called linear relations. Fix a positive integer g and an oriented Riemann surface Σ_g of genus g. For a r-tuple $\underline{d} = (d_1,\dots,d_r)$ of positive integers which fulfill the linear relations, the Hamiltonian / gauge theoretical Gromov-Witten invariant of genus g and degree $\underline{d}$ of X is a linear map

$$I^g_{\underline{d}} : A^*_T \longrightarrow \bigwedge^{\text{even}} H^1(\Sigma_g;\mathbb{Z})^{\otimes l}, \quad \text{for } l := r - \dim X.$$

Its restriction to $A_T^{|\underline{d}|-\dim X\cdot(g-1)}$ is $\mathbb{Z}$-valued.

Main results

(i) *Consider two r-tuples of integers $\underline{d}$ and $\underline{e}$ which fulfill the linear relations. Then*

$$I^g_{\underline{d}}(a) = I^g_{\underline{d}+\underline{e}}(\chi_1^{e_1}\cdot\ldots\cdot\chi_r^{e_r}\cdot a), \quad \forall a \in A^*_T.$$

(ii) *Let C be a smooth, irreducible curve of genus g, and $\underline{d} = (d_\rho)_\rho$ such that $d_\rho > 2g-1$ for $\rho = 1,\dots,r$. Put $m := |\underline{d}| - \dim X\cdot(g-1)$, and consider distinct points $\zeta_1,\dots,\zeta_m \in C$. Let $\eta_1,\dots,\eta_m \in A^1_X \cong A^1_T$ be not necessarily distinct nef classes, and take $Y_i \hookrightarrow X$ a general divisor in the linear system of η_i .Then*

$$I^g_{\underline{d}}(\eta_1\cdot\ldots\cdot\eta_m) = \#\left\{u : C\to X \;\middle|\; \begin{array}{l} \text{the degree of } u \text{ is } \underline{\mathrm{d}}, \text{ and} \\ u\,(\zeta_i)\in Y_i,\ \forall i = 1,\dots,m \end{array}\right\}$$

(iii) *There exists an equivariant class $c \in A_T^{\dim X}$ such that*

$$I^g_{\underline{d}}(a) = I^0_{\underline{d}}\big(c^g\cdot a\big), \quad \forall a \in A_T^{|\underline{d}|-\dim X\cdot(g-1)}.$$

The article is structured as follows: first we fix the notations and recall basic facts concerning toric manifolds. The study itself starts in the second section, which addresses the problem of compactifying the space of morphisms, and that of defining the virtual fundamental class for this compactification. The Hamiltonian invariants are introduced in section 5, and they are obtained by intersecting some natural cycles with the virtual fundamental class. Using localization with respect to the action of the big torus of X on the compactified space of morphisms, we obtain explicit formulae for our invariants, but we must say that in practice one needs the help of a computer for applying them because of the length of the computations which appear.

The major shortcoming of the numerical computations is that of giving no insight into the structure of the invariants; this issue is studied in the sections 7 and 8. Using standard degeneration arguments, we prove that the Hamiltonian invariants of toric varieties obey the simple recursive relations mentioned above, and therefore the computation of the higher genus invariants reduces to those of genus zero.

We conclude the article with some explicit computations, which illustrate the general theory.

2. Setting up the problem

In this section we wish to fix some basic notations; further results needed in this paper are collected in appendix A, and the references for the theory of toric varieties are [8, 13, 19]. In the whole paper $\Sigma \subset N_{\mathbb{R}} := \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{R})$, with $M \cong \mathbb{Z}^n$, denotes a projective fan and $X = X_\Sigma$ stands for the corresponding toric variety. The set of one dimensional cones of Σ is denoted $\Sigma(1)$, and we let $r := \#\Sigma(1)$ and $l := r - n$. We denote by $e^1, \dots, e^l, e^{l+1}, \dots, e^r$ the integral generators of $\Sigma(1)$. In the whole paper we will use the indices $\lambda \in \{1, \dots, l\}$, $\nu \in \{1, \dots, n\}$ and $\rho \in \{1, \dots, r\}$.

For each ρ we introduce a formal vector w_ρ; then we get the following sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow K := \ker(a) \longrightarrow \oplus_\rho \mathbb{Z} w_\rho \xrightarrow{\ a\ } N, \quad w_\rho \overset{a}{\longmapsto} e^\rho, \tag{2.1}$$

and the cokernel of a is a torsion $\mathbb{Z}$-module. By dualizing using $\mathsf{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z})$, we obtain the exact sequence

$$\begin{gathered} 0 \longrightarrow M \longrightarrow \oplus_\rho \mathbb{Z} w_\rho^\vee \xrightarrow{\ c\ } A^1(X) \longrightarrow 0, \\ m \longmapsto \big(\langle m, e^\rho \rangle\big)_\rho \quad \text{and} \quad c := \text{ the quotient map.} \end{gathered} \tag{2.2}$$

Dualizing again by $\mathsf{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{C}^*)$ we get the exact sequence of tori

$$1 \longrightarrow T \xrightarrow{\ \varepsilon\ } (\mathbb{C}^*)^r \longrightarrow S \longrightarrow 1. \tag{2.3}$$

The homomorphism ε induces a T-action on $\mathbb{C}^r$, and X is simply the quotient for this action. More precisely, it is proved in [6, theorem 2.1] that there is a T-invariant open subset $\Omega \subset \mathbb{C}^r$ whose complement $Z_X := \mathbb{C}^r \setminus \Omega$ has codimension at least two, such that X is the categorical quotient Ω/T. In fact Z_X is a union of linear subspaces of $\mathbb{C}^r$,

$$Z_X = \bigcup_{\pi} \mathbb{A}(\pi), \quad \mathbb{A}(\pi) := \{z^\rho = 0 \mid \rho \in \pi\}, \tag{2.4}$$

where π runs over the set of so-called primitive collections of Σ.

In this article we will focus on the case where X is smooth, or equivalently Σ is regular, meaning that the integral generators of any n-dimensional cone of Σ form a $\mathbb{Z}$-basis of N. In this case T acts freely on Ω and $X = \Omega/T$ is a geometric quotient. Let us assume that $(e^{l+1}, \dots, e^r)$ does generate a n-dimensional cone of Σ; then there are integers a^λ_ν such that

$$e^\lambda + \sum_{\nu=1}^{n} a^\lambda_\nu e^{l+\nu} = 0. \tag{2.5}$$

In the basis $(e^{l+1}, \dots, e^r)$, the exact sequence (2.3) takes the form

$$1 \longrightarrow (\mathbb{C}^*)^l \xrightarrow{\varepsilon} (\mathbb{C}^*)^r \longrightarrow S \longrightarrow 1, \tag{2.6}$$

with the homomorphism ε defined by the characters $(\chi_\rho)_\rho$ as follows:

$$\chi_\lambda(t) = t_\lambda \quad \text{and} \quad \chi_{l+\nu}(t) = t^{\underline{a}_\nu} = t_1^{a^1_\nu} \cdot \ldots \cdot t_l^{a^l_\nu}. \tag{2.7}$$

We should keep in mind that this last description depends on the choice of a n-dimensional cone of Σ, and this remark will be used repeatedly in the paper.

The generators $(e^\rho)_\rho$ of $\Sigma(1)$ define respectively the divisors $(D_\rho)_\rho$ on X which, as elements of $A^1(X)$, satisfy the linear relations

$$[D_{l+\nu}] - \sum_{\lambda=1}^{l} a^\lambda_\nu [D_\lambda] = 0, \text{ for all } \nu = 1, \dots, n. \tag{2.8}$$

In this way we get an isomorphism $A^1(X) \cong \mathbb{Z}[D_1] \oplus \cdots \oplus \mathbb{Z}[D_l]$ determined by the choice of the n-dimensional cone $(e^{l+1}, \dots, e^r)$ of Σ.

Definition 2.1. *Let C be a smooth and projective curve of genus g. We say that a morphism $u : C \to X$ has multi-degree $\underline{d} = (d_\rho)_\rho$ if $d_\rho = D_\rho \cdot u_* C$ for $\rho = 1, \dots, r$.*

Of course, the integers d_ρ are not independent; they are related by

$$d_{l+\nu} = \sum_{\lambda=1}^{l} a^\lambda_\nu d_\lambda \text{ for all } \nu, \text{ or equivalently } \sum_{\rho=1}^{r} d_\rho e^\rho = 0. \tag{2.9}$$

The tangent bundle of a smooth toric variety X fits into the 'Euler sequence'

$$0 \longrightarrow \mathcal{O}_X^{\oplus l} \longrightarrow \bigoplus_{\rho} \mathcal{O}_X(D_\rho) \longrightarrow \mathsf{T}_X \longrightarrow 0, \tag{2.10}$$

and together with the Riemann-Roch theorem we immediately deduce the

Lemma 2.1. *The space* $\mathsf{Mor}_{\underline{d}}(C, X)$ *of morphisms having multi-degree* $\underline{d}$ *from* C *to the smooth and projective toric variety* X *is smooth as soon as* $d_\rho > 2g - 1$ *for all* $\rho = 1, \dots, r$. *In this case, it is also irreducible and has the expected dimension*

$$\dim \mathsf{Mor}_{\underline{d}}(C, X) = \sum_{\rho=1}^{r} d_\rho - n(g-1) = |\underline{d}| - n(g-1). \tag{2.11}$$

Proof. All the statements are obvious except the one concerning the irreducibility of $\mathsf{Mor}_{\underline{d}}(C, X)$ which will be proved in corollary 3.1. □

We fix once for all a point $\zeta_0 \in C$, and consider the Poincaré bundle $\mathcal{L}_0 \to \mathcal{J} \times C$ whose restriction $\mathcal{L}_0|_{\mathcal{J}\times\{\zeta_0\}} = \mathcal{O}_{\mathcal{J}}$; for $d \in \mathbb{Z}$, we define $\mathcal{L}_d := \mathcal{L}_0 \otimes \mathrm{pr}_C^* \mathcal{O}(d\zeta_0)$. As one expects, $\mathcal{J}$ denotes the Jacobian variety of C and, for all integers d, $\mathcal{L}_d$ is a universal family for the line bundles of degree d over the curve C.

The topological type of a holomorphic principal T-bundle over C is given by its multi-degree $\underline{\delta} = (d_1, \dots, d_l)$; holomorphic principal T-bundles over C with fixed multi-degree, are parameterized by the l^{th} power of the Jacobian of C. Let $\mathcal{P}_{\underline{\delta}} \longrightarrow \mathcal{J}^l \times C$ (resp. $\mathcal{P}_{\underline{d}} \to \mathcal{J}^r \times C$) be the universal principal T-bundle (resp. $(\mathbb{C}^*)^r$-bundle), parameterizing principal T-bundles (resp. $(\mathbb{C}^*)^r$-bundles) over C with multi-degree $\underline{\delta}$ (resp. $\underline{d}$), trivialised at ζ_0. For

$$\begin{aligned} &\psi : \mathcal{J}^l \longrightarrow \mathcal{J}^r \quad \text{defined by} \\ &(L_1, \dots, L_l) \longmapsto \Big(L_1, \dots, L_l, \sum_{\lambda=1}^{l} a_1^\lambda L_\lambda, \dots, \sum_{\lambda=1}^{l} a_n^\lambda L_\lambda\Big), \end{aligned} \tag{2.12}$$

we get the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{\underline{\delta}} \times_\varepsilon (\mathbb{C}^*)^r = \psi^* \mathcal{P}_{\underline{d}} & \longrightarrow & \mathcal{P}_{\underline{d}} \\ \downarrow & & \downarrow \\ \mathcal{J}^l \times C & \xrightarrow{\psi \times \mathrm{id}_C} & \mathcal{J}^r \times C. \end{array} \tag{2.13}$$

Shortly, the reason for introducing this new ingredient is that for writing the left-hand-side of (2.13) we have chosen a cone of Σ, while the right-hand-side is symmetric, the information on the structure of Σ being encoded in the map ψ. The image of ψ can be intrinsically described as

$$G_C := \mathrm{Image}(\psi) = \big\{(x_1, \dots, x_r) \in \mathcal{J}^r \mid \langle m, e^\rho \rangle x_\rho = 0 \in \mathcal{J},\ \forall m \in M\big\}.$$

3. Description of the space of morphisms

All the subsequent constructions are motivated by the following very simple remark: given a morphism $u : C \to X$ having multi-degree $\underline{d}$, the pull-back $P := u^*\Omega \to C$ is a holomorphic principal T-bundle whose multi-degree is $\underline{\delta}$. The morphism $C = P/T \to P \times_T \mathbb{C}^r$ is just a section of a rank r vector bundle over C on which the torus T still acts, covering the identity of C. Any two sections which are in the same T-orbit give rise to the same morphism from C into X (some care is actually required at this point). This is the correspondence which will be exploited in this section, and one easily recognizes in it the functorial description of a toric variety, as is presented in [7].

Let us assume for the beginning that the inequalities appearing in lemma 2.1 hold. We start with the vector bundles associated to (2.13)

$$\begin{array}{ccc} \mathcal{P}_{\underline{\delta}} \times_T \mathbb{C}^r = \psi^*(\mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r) & \longrightarrow & \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{C}^r = \oplus_\rho \mathcal{L}_\rho \\ \downarrow & & \downarrow \\ \mathcal{J}^l \times C & \xrightarrow{\psi \times \mathrm{id}_C} & \mathcal{J}^r \times C, \end{array} \tag{3.1}$$

and notice that $\psi^*\mathcal{L}_\rho = \mathcal{P}_{\underline{\delta}} \times_{\chi_\rho} \mathbb{C}$ (the characters χ_ρ are defined by (2.7)). Taking the direct images under $p : \mathcal{J}^r \times C \to \mathcal{J}^r$ gives

$$\begin{array}{ccc} \mathcal{V} = \oplus_\rho \mathcal{V}_\rho := \mathcal{W}|_{G_C} & \longrightarrow & \mathcal{W} = \oplus_\rho \mathcal{W}_\rho := p_*(\oplus_\rho \mathcal{L}_\rho) \\ \downarrow & & \downarrow Q \\ G_C & \overset{J}{\hookrightarrow} & \mathcal{J}^r, \end{array} \tag{3.2}$$

and we recognize in $\mathcal{W}_\rho$ the Picard vector bundles associated respectively to the Poincaré bundles $\mathcal{L}_\rho$; the rank of $\mathcal{W}$ is given by the formula

$$\operatorname{rk} \mathcal{W} = \sum_{\rho=1}^{r} d_\rho - r(g-1) = |\underline{d}| - r(g-1).$$

The action of T on $\mathbb{C}^r$ induces actions on $\oplus_\rho \psi^*\mathcal{L}_\rho$ and $\oplus_\rho \mathcal{L}_\rho$ covering respectively the identities of $\mathcal{J}^l \times C$ and $\mathcal{J}^r \times C$ and, *a fortiori*, there are natural T-actions on $\mathcal{V}$ and $\mathcal{W}$ which cover respectively the identities of $\mathcal{J}^l$ and $\mathcal{J}^r$ and moreover preserve the decompositions $\mathcal{V} = \oplus_\rho \mathcal{V}_\rho$ and $\mathcal{W} = \oplus_\rho \mathcal{W}_\rho$.

By our previous remark, the space of morphisms from C to X should be the quotient '$\mathcal{V}/T$'. Of course, this statement must not be taken *ad litteram* but in the spirit of geometric invariant theory. What we shall actually construct is the invariant quotient of $\mathcal{W}$ for the T-action, and $\mathcal{V}/\!\!/T$ will be its restriction to G_C.

One can spot at the first glance a 'nice' Zariski open subset of $\mathcal{W}$ on which T acts freely

$$\mathcal{W}^o := \{s \in \mathcal{W} \mid \operatorname{Image} s \not\subset \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} Z_X\}. \tag{3.3}$$

The subvariety $Z_X \subset \mathbb{C}^r$ which had to be 'thrown away' for obtaining X was a union of coordinate subspaces, and therefore

$$Z_W := \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} Z_X\big) = \bigcup_\pi \Gamma\big(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \mathbb{A}(\pi)\big) \tag{3.4}$$

is still a union of subvector bundles of $\mathcal{W}$, and $\mathcal{W}^o = \mathcal{W} \setminus Z_W$. Though T is acting freely on $\mathcal{W}^o$, it is possibly not so clear that the quotient $\mathcal{W}^o/T$ exists.

Proposition 3.1. *The T-action on $\mathcal{W}$ can be linearized in a line bundle over it such that the T-stable set for this action coincides with $\mathcal{W}^o$ defined by* (3.3). *Moreover, the invariant quotient $W := \mathcal{W}/\!\!/T$ is a smooth and projective variety of dimension*

$$\dim W = \sum_{\rho=1}^{r} d_\rho + n.$$

The natural projection $q : W \to \mathfrak{J}^r$ is a locally trivial fibration whose fibres are isomorphic to a smooth and projective toric variety X'.

Proof. According to corollary A.1, if the T-action on $\mathbb{C}^r$ is linearized in the trivial line bundle $A \to \mathbb{C}^r$ using some ample class $\beta \in A^1(X)$, the corresponding T-stable set is Ω and $X = \mathbb{C}^r /\!\!/T$. We recognize now that we are in the situation studied in [15, theorem 2.3], where is proved that fibrewise, over each $j \in \mathfrak{J}^r$, the quotient $\mathcal{W}^o_j/T$ exists, is smooth and projective, and actually coincides with the invariant quotient $\mathcal{W}_j/\!\!/T$ for the T-action linearized in $\mathcal{O}(\mathcal{W}_j)$ using the same character β as before.

We choose an ample line bundle $\mathcal{A} \to \mathfrak{J}^r$ such that $\mathcal{A} \otimes \big(Q_*\mathcal{O}(\mathcal{W})\big)^T$ is globally generated, where $Q : \mathcal{W} \to \mathfrak{J}^r$ is the natural projection. We linearize the T-action on $\mathcal{W}$ in $Q^*\mathcal{A}$ using the character β, and we claim that the corresponding T-stable locus is precisely $\mathcal{W}^o$. That amounts to the possibility of extending T-equivariant sections of $\mathcal{O}(\mathcal{W}_j)$, $j \in \mathfrak{J}^r$, to global T-equivariant sections in $Q^*\mathcal{A}$, and this is ensured by the choice of a sufficiently ample $\mathcal{A}$. □

The precise relationship between the toric varieties X and X' is described in detail in [15, section 5]. What is relevant for our study, and will be used in proposition 4.1, is that $A^1(X') \cong A^1(X)$.

We recall now that what we are actually interested in is a compactification of the space $\mathsf{Mor}_{\underline{d}}(C, X)$. The reason for introducing the variety W was to have a 'symmetric object' in our hands, in the sense that it does not depend on the choice of some particular cone of Σ. The compactification we are looking for is $V_{\underline{d}}(C) := \mathcal{V}/\!\!/T$ (see (3.2)), which coincides with $\psi^*W = \mathfrak{J}^l \times_{\mathfrak{J}^r} W$. It is a locally trivial fibre bundle over $\mathfrak{J}^l$, with the fibres isomorphic to the toric variety X'. We collect this information in

Corollary 3.1. *The space of morphisms $\mathsf{Mor}_{\underline{d}}(C, X)$ is irreducible and the variety $V_{\underline{d}}(C) := W|_{G_C}$ is a smooth and projective compactification of it.*

Recall that we have imposed at the beginning of the section the inequalities $d_\rho > 2g - 1$, for $\rho = 1, \ldots, r$. We are going to remove now this restriction and describe the spaces of morphisms for arbitrary multi-degree $\underline{d}$. In such cases, the moduli space itself is usually oversized, but still carries a homology class of right dimension, which replaces the fundamental class. Such an object is usually called in the literature a *virtual fundamental class*. For our purposes it will be enough to apply Fulton's construction of the localized Euler class. This approach has been already used in the context of gauge theoretical Gromov-Witten invariants in [23, section 4.1].

So, let us fix a collection of integers $\underline{d} = (d_\rho)_\rho$ satisfying the linear relations (2.9). We *choose* now another set of integers $\underline{d}' = (d'_\rho)_\rho$ satisfying the same linear relations, and moreover

$$d'_\rho > 2g - 1 \quad \text{and} \quad e_\rho := d'_\rho - d_\rho > 0, \quad \forall \rho. \tag{3.5}$$

Using the canonical sections in $\mathcal{O}_C(e_\rho \zeta_0)$ we get the exact sequence of sheaves

$$0 \longrightarrow \bigoplus_\rho \omega_C \otimes \mathcal{L}_{-d'_\rho} \longrightarrow \bigoplus_\rho \omega_C \otimes \mathcal{L}_{-d_\rho} \longrightarrow \bigoplus_\rho \mathcal{O}_{\mathcal{J}}^{\oplus e_\rho} \longrightarrow 0$$

over $\mathcal{J}^r \times C$, and we consider the direct image on $\mathcal{J}^r$

$$\begin{aligned} 0 \ &\to \underset{\rho}{\oplus} p_*(\omega_C \otimes \mathcal{L}_{-d_\rho}) \to \underset{\rho}{\oplus} \mathcal{O}_{\mathcal{J}}^{\oplus e_\rho} \to \underset{\rho}{\oplus} R^1 p_*(\omega_C \otimes \mathcal{L}_{-d'_\rho}) \\ &\to \underset{\rho}{\oplus} R^1 p_*(\omega_C \otimes \mathcal{L}_{-d_\rho}) \to 0. \end{aligned}$$

The last epimorphism induces the closed embedding between the associated linear fibre spaces

$$\begin{aligned} \mathcal{W} &:= \mathsf{Spec}\big[\mathsf{Sym}^\bullet \underset{\rho}{\oplus} R^1 p_*(\omega_C \otimes \mathcal{L}_{-d_\rho})\big] \hookrightarrow \\ \mathcal{W}' &:= \mathsf{Spec}\big[\mathsf{Sym}^\bullet \underset{\rho}{\oplus} R^1 p_*(\omega_C \otimes \mathcal{L}_{-d'_\rho})\big]. \end{aligned}$$

This construction is made in such a way that the fibre of $\mathcal{W}$ over a point $j = (j_\rho)_\rho \in \mathcal{J}^r$ is precisely $\oplus_\rho H^0(C, \mathcal{L}_{d_\rho, j_\rho}) \subset \oplus_\rho H^0(C, \mathcal{L}_{d'_\rho, j_\rho})$. Moreover, there is a natural T-equivariant evaluation morphism $\mathcal{W}' \xrightarrow{\upsilon} \oplus_\rho \mathbb{C}^{\oplus e_\rho}$, and $\mathcal{W}$ coincides with its vanishing locus. We define $\mathcal{V}_\rho := \mathcal{W}_\rho|_{G_C}$: it has the structure of a *linear fibre space* over G_C, but does not need to be a vector bundle anymore. Using the notation (3.3), we observe that $\mathcal{W}^o = (\mathcal{W}')^o \cap \mathcal{W}$, and therefore

$$W_{\underline{d}} := \mathcal{W}^o / T \hookrightarrow (\mathcal{W}')^o / T =: W_{\underline{d}'},$$

is a closed embedding; more precisely, $W_{\underline{d}}$ is the zero locus of the section

$$\hat{\upsilon} : W_{\underline{d}'} \longrightarrow (\mathcal{W}')^o \times_T \big(\oplus_\rho \mathbb{C}^{\oplus e_\rho}\big),$$

induced by υ. Using the morphism ψ, we get the Cartesian diagram

$$\begin{tikzcd} V_{\underline{d}}(C) \arrow[r, hook, "\iota"] \arrow[d] & V_{\underline{d}'}(C) \arrow[d] \\ W_{\underline{d}} \arrow[r, hook] & W_{\underline{d}'} \end{tikzcd} \tag{3.6}$$

and $V_{\underline{d}}(C) \hookrightarrow V_{\underline{d}'}(C)$ is the zero locus of $\psi^* \hat{\upsilon}$. According to [12, section 6.2], in this case we have the refined Gysin homomorphism

$$0^! : A_*\big(V_{\underline{d}'}(C)\big) \longrightarrow A_{*-|\underline{e}|}\big(V_{\underline{d}}(C)\big),$$

and we define the *virtual fundamental class* as

$$[\![V_{\underline{d}}(C)]\!] := 0^![V_{\underline{d}'}(C)] \in A_{|\underline{d}|-n(g-1)}\big(V_{\underline{d}}(C)\big).$$

It has the property that

$$\imath_*[\![V_{\underline{d}}(C)]\!] = [V_{\underline{d}'}(C)] \cap c_{\mathrm{top}}\big((\mathcal{V}')^o \times_T \oplus_\rho \mathbb{C}^{\oplus e_\rho}\big). \tag{3.7}$$

When $V_{\underline{d}}(C)$ has the expected dimension, this construction yields the usual fundamental class.

Finally, we remark that the class $[\![V_{\underline{d}}(C)]\!]$, constructed as above, is independent on the choice of the multi-degree $\underline{d}'$. This is an immediate consequence of the associativity property of the refined Gysin homomorphism (see [12, theorem 6.5]). The intrinsic nature of $[\![V_{\underline{d}}(C)]\!]$ follows also from [23, theorem 5.10], where the compactification $V_{\underline{d}}(C)$ is obtained using gauge theoretical methods.

We wish to clarify now our statement in the introduction, that $V_{\underline{d}}(C)$ differs from the stable map compactification of the space of morphisms from C to X, and it is only birational to an irreducible component of this latter one. Indeed, the space of stable maps whose stabilized domain is C contains, for $g \geq 2$, the component whose points correspond to the following morphisms: the domain of definition is the singular curve consisting of C with $\mathbb{P}^1$ attached at some point; the map is constant on C and has multi-degree $\underline{d}$ on $\mathbb{P}^1$.

Another fundamental difference between the two compactifications is that, in contrast to the Kontsevich-Manin compactification whose closed points correspond to stable maps, not all the closed points of $V_{\underline{d}}(C)$ can be identified with a stable map. Indeed, let us assume again that the components of $\underline{d}$ are $d_\rho > 2g-1$, and for a primitive collection $\pi \subset \Sigma(1)$ let us define the subvariety

$$V_{\underline{d},\pi}(C) := \big\{[(s_\rho)_\rho] \mid \exists \zeta \in C \text{ s.t. } s_\rho(\zeta) = 0,\ \forall \rho \in \pi\big\} \hookrightarrow V_{\underline{d}}(C). \tag{3.8}$$

Its dimension is obviously $|\underline{d}| - n(g-1) - (|\pi| - 1)$. However, the multi-degree of the morphisms $C \to X$ defined by such tuples of sections (using the evaluative criterion of properness for X) is componentwise smaller than $\underline{d}$, with *strict* inequality for $\rho \in \pi$. We can say more precisely

Lemma 3.1. *Assume that $\underline{d} = (d_\rho)_\rho$ satisfies $d_\rho > 2g-1$ for all ρ. Then the generic multi-degree of the morphisms induced by the points of $V_{\underline{d},\pi}(C)$ equals $\underline{d} - \underline{d}_\pi$ (see remark A.2 for the definition of $\underline{d}_\pi$).*

Proof. We fix a strictly convex Σ-linear function η on $N_{\mathbb{R}}$; this corresponds to the choice of an ample divisor on X. Let us fix a point $\zeta \in C$, and consider a general point $[(s_\rho)_\rho] \in V_{\underline{d},\pi}(C)$ such that the $\{s_\rho\}_{\rho\in\pi}$'s vanish at ζ. The multi-degree of the corresponding morphism $v_{\underline{s}} : C \to X$ equals $\underline{d} - \underline{d}'$, where $\underline{d}' = (d'_\rho)_\rho$ obeys (2.9), and $d'_\rho \geq 0$ for $\rho \in \Sigma(1)$, with strict inequality for $\rho \in \pi$. It follows that

$$-\sum_{\rho\in\pi} e^\rho = \sum_{\rho\in\Sigma(1)\setminus\pi} d'_\rho e^\rho + \sum_{\rho\in\pi}(d'_\rho - 1)e^\rho,$$

and therefore

$$\eta\Big(-\sum_{\rho\in\pi} e^{\rho}\Big) \leq \sum_{\rho\in\Sigma(1)\setminus\pi} d'_{\rho}\eta(e^{\rho}) + \sum_{\rho\in\pi}(d'_{\rho}-1)\eta(e^{\rho}) =: \sum_{\rho\in\Sigma(1)} d''_{\rho}\eta(e^{\rho}).$$

We rewrite the last inequality as

$$\sum_{\rho\in\pi}\eta(e^{\rho}) + \eta\Big(-\sum_{\rho\in\pi} e^{\rho}\Big) \leq \langle \eta, \underline{d}'\rangle.$$

This means that the 'energy loss' with respect to the ample divisor η is bounded below. We recall now that we choose $[\underline{s}] = [(s_{\rho})_{\rho}]$ general, which implies that $v_{\underline{s}}$ looses the minimal possible amount of energy, or – in terms of sections – that the components s_{ρ} vanish at the minimal number of points. Consequently, the inequality above is equality. Since η is strictly convex, the set $\{e^{\rho} \mid d''_{\rho} > 0\}$ must be contained in a cone of Σ, so that $\underline{d}' = \underline{d}_{\pi}$, by the very definition of $\underline{d}_{\pi}$. □

In this case we clearly see that dim $V_{\underline{d}-\underline{d}_{\pi}}(C) <$ dim $V_{\underline{d},\pi}(C)$, and the reason is that in this compactification 'we forget' about the bubble components, but we still keep track of the bubbling points.

4. Cohomology of the space of morphisms

Now that we have a good description for the compactification of the space of morphisms, we wish to compute its cohomology ring. We will assume again that the multi-degree $\underline{d}$ of the maps satisfies $d_{\rho} > 2g-1$ for all ρ. The objects which will play a central role in our study are the line bundles

$$\Lambda_{\rho} := \mathcal{W}^{o} \times_{\chi_{\rho}} \mathbb{C} \longrightarrow W, \quad \forall \rho = 1, \ldots, r \tag{4.1}$$

on $W_{\underline{d}}$ and their pull-back to $V_{\underline{d}}(C) \overset{J}{\hookrightarrow} W_{\underline{d}}$.

Obviously, the classes Λ_{ρ} satisfy the linear relations as

$$\Lambda_{l+\nu} = \sum_{\lambda=1}^{l} a_{\nu}^{\lambda}\Lambda_{\lambda}, \quad \forall \nu \in \{1, \ldots, n\}, \tag{4.2}$$

corresponding to the n-dimensional cones of Σ. They also satisfy a set of non-linear relations, that we are going to describe. We observe that for each ρ there is a *sheaf* monomorphism

$$0 \to \Lambda_{\rho}^{-1} \to q^{*}\mathcal{W}_{\rho} \quad \text{given by} \quad [s, z] \longmapsto z\,\mathrm{pr}_{\mathcal{W}_{\rho}} s,\ \forall s \in \mathcal{W}^{o} \text{ and } \forall z \in \mathbb{C}.$$

Equivalently, one can say that for each ρ there is a canonical non-zero section $0 \to \mathcal{O}_W \to q^{*}\mathcal{W}_{\rho} \otimes \Lambda_{\rho}$. As $W = (\mathcal{W} \setminus Z_W)/T$, with Z_W defined by (3.4), we deduce that for every primitive collection π,

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \bigoplus_{\rho\in\pi} q^{*}\mathcal{W}_{\rho} \otimes \Lambda_{\rho} \tag{4.3}$$

is a monomorphism of vector bundles, so that the Euler class

$$e\big(\oplus_{\rho\in\pi} q^*\mathcal{W}_\rho \otimes \Lambda_\rho\big) = 0. \tag{4.4}$$

When X is a projective space, so that $W = \mathbb{P}(\mathcal{W})$, this equality reduces to the standard Grothendieck relation for $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1) \to W$.

Proposition 4.1. *The cohomology and the Chow rings of $W_{\underline{d}}$ (resp. $V_{\underline{d}}(C)$) are:*

$$\begin{aligned}
&H^*(W_{\underline{d}};\mathbb{Z}) \cong H^*(\mathcal{J}^r;\mathbb{Z})[\xi_1,\dots,\xi_r]\big/\langle\mathcal{LR}\rangle + \langle\mathcal{SR}\rangle,\\
&A^*(W_{\underline{d}}) \cong A^*(\mathcal{J}^r)[\xi_1,\dots,\xi_r]\big/\langle\mathcal{LR}\rangle + \langle\mathcal{SR}\rangle,\\
&H^*(V_{\underline{d}}(C);\mathbb{Z}) \cong H^*(G_C;\mathbb{Z})[\xi_1,\dots,\xi_r]\big/\jmath^*\langle\mathcal{LR}\rangle + \jmath^*\langle\mathcal{SR}\rangle,\\
&A^*(V_{\underline{d}}(C)) \cong A^*(G_C)[\xi_1,\dots,\xi_r]\big/\jmath^*\langle\mathcal{LR}\rangle + \jmath^*\langle\mathcal{SR}\rangle.
\end{aligned}$$

In these formulae, $\xi_1,\dots,\xi_r$ are formal variables, and $\langle\mathcal{LR}\rangle$ (resp. $\langle\mathcal{SR}\rangle$) are the ideals generated by (4.2) *(resp.* (4.4)*), obtained by formally replacing Λ_ρ with ξ_ρ.*

Proof. The statement concerning the Chow ring is proved in [15, theorem 4.2], while that about the cohomology ring is proved in [15, theorem 3.9]. □

Remark 4.1. We notice that the Chow ring of X' gives approximations, in the sense of [9], for the T-equivariant Chow ring of a point. According to [9, definition-proposition 1],

$$A^i_T \cong A^i(X'), \text{ as soon as } \operatorname{codim} Z_W \geq i.$$

What will be of interest for us is that $A^n(X') \cong A^n_T$ for multi-degrees $\underline{d} = (d_\rho)_\rho$ with $d_\rho \geq \max\big\{\frac{n}{2} + g - 1, 2g\big\}$ for all ρ.

For arbitrary multi-degrees we must be more careful: since we have little control on what $V_{\underline{d}}(C)$ looks like, we can not describe its Chow ring. However, the line bundles $\Lambda_\rho \to V_{\underline{d}}(C)$ still exist, because $\mathcal{V}^o \to V_{\underline{d}}(C)$ is a principal bundle (in the étale topology); this latter is just the restriction of $(\mathcal{V}')^o \to V_{\underline{d}'}(C)$ to the closed subset $V_{\underline{d}}(C) \hookrightarrow V_{\underline{d}'}(C)$. Then, according to [12, example 17.1.1], Λ_ρ defines an element in $A^1\big(V_{\underline{d}}(C)\big)$, but we must be aware that in this case the operational Chow group is not necessarily Poincaré dual to the usual Chow group. This operational class can also be defined as the pull-back $\imath^*[\Lambda_\rho]$, for $[\Lambda_\rho] \in A^1\big(V_{\underline{d}'}(C)\big)$, and one observes again that this definition is independent of the choice of $\underline{d}'$.

5. The invariants

There is yet another reason why $\Lambda_1,\dots,\Lambda_r$ introduced in the previous section are of interest: at least for $\underline{d}$ suitably large, the $\psi^*\Lambda_\rho$'s coincide respectively with the restriction to the fibres of $V_{\underline{d}}(C) \times C \to C$ of the pull-backs of the line bundles $\mathcal{O}_X(D_\rho)$, under the *rational* evaluation map

$$ev : V_{\underline{d}}(C) \times C \dashrightarrow X, \tag{5.1}$$

at least on the domain of definition of ev. Since the classes $[D_\rho]$ generate the Chow group of X, we may hope that integrals such as $\int_{V_{\underline{d}}(C)} \prod_\rho (\psi^*\Lambda_\rho)^{m_\rho}$ are related to enumerative invariants of X. Morally, they should count the number of morphisms from C to X satisfying certain incidence conditions.

The unpleasant feature of this description is that the evaluation above is not everywhere defined, and therefore we are forced to use an equivariant version of it. Let us consider again the universal principal T-bundle $\mathcal{P}_{\underline{\delta}}$ introduced in the first section, and $\mathcal{V}^o := \psi^*\mathcal{W}^o$. We recall that any section $s \in \Gamma(C, \mathcal{P}_{\underline{\delta},j} \times_\varepsilon \mathbb{C}^r)$, $j \in \mathcal{J}^l$, can be interpreted as a T-equivariant morphism $u_s : \mathcal{P}_j \to \mathbb{C}^r$; here we use the notation $\mathcal{P}_j$ for the restriction $\mathcal{P}|_{\{j\}\times C}$. Consequently there is a well-defined evaluation morphism

$$\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o \longrightarrow \mathbb{C}^r.$$

The product $T \times T$ acts on $\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o$ by

$$(t, \tau) \times (p, s) := (\mathrm{R}_{t^{-1}\tau} p, t \times s),$$

and the evaluation above is invariant for the action of the first T-factor, while it is T-equivariant for the action of the second factor. Overall the capital letter ‘R’ denotes the right T-action on the principal bundle $\mathcal{P}_{\underline{\delta}}$. On the quotient $\big(\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T$ corresponding to the action of *the first* T-factor, we obtain an induced evaluation morphism

$$\Phi : \big(\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T \longrightarrow \mathbb{C}^r, \tag{5.2}$$

which is equivariant for the remaining T-action. With the notations of (3.2), Φ is simply

$$\bigoplus_\rho \mathcal{L}_\rho^{-1} \otimes \mathcal{V}_\rho \longrightarrow \mathbb{C}^r.$$

Therefore we have an induced ring homomorphism between the equivariant Chow rings

$$\Phi_T^* : A_T^* \cong A_T^*(\mathbb{C}^r) \longrightarrow A_T^*\left(\big(\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T\right) \cong A^*(V_{\underline{d}}(C) \times C),$$

where A_T^* denotes the T-equivariant Chow ring of a point. The last isomorphism follows from [9, theorem 4], because the action of T on $\mathcal{V}^o$ is proper: indeed we can always embed it into some $(\mathcal{V}')^o$, and this latter is, according to proposition 3.1, the T-stable locus in $\mathcal{V}'$.

The Chow ring $A_T^* \cong \mathsf{Sym}^\bullet\big(\mathcal{X}^*(T)\big) \cong \mathbb{Z}[D_1, \ldots, D_r]/\langle$linear relations’ ideal$\rangle$, and we observe that $\Phi_T^*[D_\rho]/[pt] = [\Lambda_\rho] \in A^*(V_{\underline{d}}(C))$. By the slant product we mean the following: pick an arbitrary point $pt \in C$, and define $\Phi_T^*(a)/[pt] := \jmath_{pt}^*\Phi_T^*(a) \in A^*(V)$, for $\jmath_{pt} : V_{\underline{d}}(C) \hookrightarrow C \times V_{\underline{d}}(C)$ the corresponding inclusion. We remark that the slant product, and the equality itself makes sense only modulo homological equivalence. We will always write $\chi_\rho := [D_\rho] \in A_T^*$.

We must say that this evaluation map has already appeared in [22, page 553] and [5, page 586] in the gauge theoretical setting, and that the formulae introduced in *loc. cit.* take this down-to-earth form in the case of toric varieties.

Definition 5.1. *The Hamiltonian / gauge theoretical invariant of genus g and multi-degree* $\underline{d} = (d_1, \ldots, d_r)$ *of X is the homomorphism*

$$I_{\underline{d}}^g : A_T^* \to H_*(G_C) \cong H_*(\mathfrak{J}^l) \quad \textit{defined by}$$

$$I_{\underline{d}}^g(a) := (\mathrm{cl}_{G_C} \circ q_*)\Big[(\Phi_T^*(a)/[pt]) \cap [\![V_{\underline{d}}(C)]\!]\Big],$$

where $\mathrm{cl}_{G_C} : A_*(G_C) \to H_*(G_C)$ *denotes the cycle homomorphism (see [12, section 19.1]). Equivalently, one defines*

$$I_{\underline{d}}^g : A^*(G_C) \otimes A_T^* \to \mathbb{Z},$$

$$I_{\underline{d}}^g(\gamma \otimes a) := \mathrm{length}\big[\gamma \cdot q_*\big((\Phi_T^*(a)/[pt]) \cap [\![V_{\underline{d}}(C)]\!]\big)\big].$$

Since A_T^* *is generated by* $\chi_1, \ldots, \chi_r$*, any such invariant is a linear combination of*

$$I_{\underline{d}}^g(m_1, \ldots, m_r) := I_{\underline{d}}^g(\chi_1^{m_1} \cdot \ldots \cdot \chi_r^{m_r}).$$

Let us make some comments about this definition. The reason for composing the proper push-forward q_* with the cycle map (respectively, to take the length of the intersection product) is to obtain quantities which are independent of C, when is viewed as a point in the Deligne-Mumford space of curves of genus g (see proposition 5.1). This will eventually allow us to deduce recursive formulae for the invariants, by deforming C to a nodal curve.

Secondly, the restriction of $I_{\underline{d}}^g$ to $A_T^{|\underline{d}|-n(g-1)}$ is $\mathbb{Z}$-valued, and the corresponding invariants have a down-to-earth enumerative meaning for X (see proposition 5.2).

Another remark is that in order to define the invariants we must use the *virtual fundamental class* of the compactification $V_{\underline{d}}(C)$ defined by (3.7). As we have pointed already out, if the inequalities $d_\rho > 2g - 1$ are satisfied for all ρ, then $V_{\underline{d}}(C)$ has the expected dimension, and the virtual fundamental class coincides with the usual fundamental class of $V_{\underline{d}}(C)$. We should also notice that in order to compute the invariants above, it is enough to deal with the smooth case: indeed, for an embedding of $V_{\underline{d}}(C)$ into a smooth $V_{\underline{d}'}(C)$, equality (3.7) implies that

$$[\![V_{\underline{d}}(C)]\!] = [V_{\underline{d}'}(C)] \cap \Big[\prod_\rho \Lambda_\rho^{e_\rho}\Big],$$

and therefore

$$I_{\underline{d}}^g(a) = I_{\underline{d}+\underline{e}}^g\big(\chi_1^{e_1} \cdot \ldots \cdot \chi_r^{e_r} \cdot a\big), \quad \forall a \in A_T^*. \tag{5.3}$$

Finally, we notice that there is some trouble with our notation for the invariants, since it involves only the genus g of the curve C. We are going to prove that this is indeed the case: the quantities defined above are independent of the conformal structure of C. In other words, the invariants have a topological nature in the sense that they depend only on the oriented Riemann surface underlying the projective curve C.

Proposition 5.1. *The invariant*

$$I^C_{\underline{d}}(m_1,\dots,m_r) := (\mathrm{cl}_{G_C}\circ q_*)\Big[\big(\Phi^*_T(a)/[pt]\big)\cap [\![V_{\underline{d}}(C)]\!]\Big]$$

does not depend on the conformal structure of the curve C.

Proof. By our previous discussion, it is enough to prove the statement in the smooth case. Let us consider a flat family $\mathscr{C}\to\triangle$ of smooth and irreducible curves, parameterized by an irreducible base $\triangle$, whose fibre over $o\in\triangle$ is our curve C. After replacing $\triangle$ with some open neighborhood of o and after étale base change, we may assume that there exists a section $\sigma:\triangle\to\mathscr{C}$. According to [1, theorem 3.4], in this situation the relative σ-rigidified Picard functor is representable, that is it exists the relative Jacobian $\mathscr{J}\to\triangle$ which parameterizes line bundles of degree 0 over the fibres of $\mathscr{C}/\triangle$, and it exists the Poincaré bundle $\mathscr{L}_0\to\mathscr{J}\times_\triangle\mathscr{C}$ which is trivialised along $\mathscr{J}\times_\triangle\sigma(\triangle)$. For $d\in\mathbb{Z}$, we will write $\mathscr{L}_d:=\mathscr{L}_0\otimes\mathcal{O}_{\mathscr{J}\times_\triangle\mathscr{C}}\big(\mathscr{J}\times_\triangle\sigma(\triangle)\big)$, which is a Poincaré bundle for line bundles of degree d over $\mathscr{C}/\triangle$.

Let us fix again a multi-degree $\underline{d}=(d_1,\dots,d_r)$ with $d_\rho>2g-1$, for all $\rho=1,\dots,r$. Then the relative Picard sheaf $\mathscr{W}_\rho:=p_*\mathscr{L}_\rho\to\mathscr{J}$ is locally free for all ρ, and we define

$$\mathscr{W}:=\oplus_\rho\mathscr{W}_\rho\to\underbrace{\mathscr{J}\times_\triangle\ldots\times_\triangle\mathscr{J}}_{r\text{ times}}=:\mathscr{J}^r.$$

For each primitive collection $\pi\subset\{1,\dots,r\}$ of the fan defining X, we consider $\mathscr{W}(\pi):=\ker(\mathscr{W}\to\oplus_{\rho\in\pi}\mathscr{W}_\rho)$ and let

$$\mathscr{Z}_W:=\bigcup_{\pi\text{ primitive}}\mathscr{W}(\pi),\quad \mathscr{W}^o:=\mathscr{W}\setminus\mathscr{Z}_W.$$

Proposition 3.1 implies that the quotient $\mathscr{W}^0/T$ exists, and is projective over $\triangle$.

The morphism ψ defined in (2.12) fits into a family of morphisms $\mathscr{J}^l\to\mathscr{J}^r$ over $\triangle$ which we still denote by ψ, and let $\mathscr{V}^o/T\to\triangle$ be the corresponding pull-back (fibre product). The fibre of $\mathscr{V}^o/T$ over a point $t\in\triangle$ is, according to corollary 3.1, just the compactification $V_{\underline{d}}(C_t)$ of $\mathsf{Mor}_{\underline{d}}(C_t,X)$. Moreover, for each $\rho=1,\dots,r$,

$$\mathbf{\Lambda}_\rho:=\mathscr{W}^o\times_{\chi_\rho}\mathbb{C}$$

is a line bundle over $\mathscr{W}^o/T$, whose restriction $\mathbf{\Lambda}_\rho|_{V_{\underline{d}}(C_t)}$ is the line bundle $\Lambda_\rho(t)$. Therefore the homology class $I^{C_t}_{\underline{d}}(m_1,\dots,m_r)$ is independent of $t\in\triangle$.

The Teichmüller space of curves of genus g being irreducible, the integrals above depend only on the oriented Riemann surface underlying the curve C. □

Remark 5.1. (i) The comparison theorem [23, theorem 5.10] between the algebraic geometric and the gauge theoretical virtual fundamental class (in the sense of Brussee) allows to identify the invariants defined algebraically as above with those defined using gauge-theoretical methods. In this latter case, the independence of the invariants on the conformal structure of the curve is obvious.

More precisely, let us denote by Σ_g an oriented Riemann surface of genus g, by ω_{can} and J_0 respectively the standard Kähler form and the standard complex structure on $\mathbb{C}^r$, and by $\mathfrak{m}_{\mathrm{can}} : \mathbb{C}^r \to \mathbb{R}^l \cong \mathcal{L}ie(T_{\mathbb{R}})$ the moment map corresponding to the $T_{\mathbb{R}}$-action, normalized by the condition that $\mathfrak{m}_{\mathrm{can}}(0) = 0$. Further, we choose $t_0 \in \mathbb{R}^l$ such that $X \cong \mathfrak{m}_{\mathrm{can}}^{-1}(t_0)/T_{\mathbb{R}}$.

The cited theorem implies that: $I^g_{\underline{d}}$ coincides with the gauge theoretical invariant associated to *the Hamiltonian and J_0-holomorphic action* of the maximal compact subgroup $T_{\mathbb{R}} \subset T$ on $(\mathbb{C}^r, \omega_{\mathrm{can}}, J_0)$, *the pair* $(\Sigma_g \times S_{\mathbb{R}}, \text{triv. conn.})$ consisting of the trivial principal $S_{\mathbb{R}}$-bundle over Σ_g endowed with the trivial connection, *the equivariant class* $\underline{\delta} = (d_1, \ldots, d_l) \in H^*_{T_{\mathbb{R}}}(\mathbb{C}^r)$, and t_0.

(ii) In [5, section 5], the Hamiltonian invariants are defined using *generic*, $T_{\mathbb{R}}$-invariant Hamiltonian perturbations and almost complex structures, which agree with J_0 outside some (arbitrarily large) ball in $\mathbb{C}^r$; we denote this latter space by $\mathcal{J}_0$. This makes less clear how do the gauge theoretical invariants relate to the Hamiltonian ones.

Claim. In the abelian situation described above, the Hamiltonian and the gauge theoretical invariants coincide.

Proof. We denote by $P \to \Sigma_g$ the principal $T_{\mathbb{R}}$-bundle whose topological type is $\underline{\delta}$ (see section 2), and we define $W^{k,2}_{T_{\mathbb{R}}}(P, \mathbb{C}^r; \underline{\delta})^*$ to be space consisting of those $T_{\mathbb{R}}$-equivariant maps $P \to \mathbb{C}^r$ of class L^2_k whose image is not contained in the closed subset of $\mathbb{C}^r$ consisting of points with non-trivial stabilizer. Further, we denote by $\mathscr{A}^{k,2}(P)$ the space of L^2_k-connections in P, and by $\mathscr{G}^{k+1,2}(P)$ the space of gauge transformations of P, of class L^2_{k+1}. We define

$$\mathcal{B}^* := \frac{W^{k,2}_{T_{\mathbb{R}}}(P, \mathbb{C}^r; \underline{\delta})^* \times \mathscr{A}^{k,2}(P)}{\mathscr{G}^{k+1,2}(P)}.$$

All these spaces naturally possess the structure of Hilbert manifolds. Observe that $\mathcal{B}^*$ is a Hilbert manifold precisely because G acts freely on $W^{k,2}_{T_{\mathbb{R}}}(P, \mathbb{C}^r; \underline{\delta})^*$.

Let $J \in \mathcal{J}_0$ and $\tau_0 := -i(t_0 + 2\pi\underline{\delta})$. The gauge equivalence classes of solutions to the unperturbed (*i. e.* the Hamiltonian perturbation vanishes) (J, τ_0)-vortex equations are realized as the zero locus of a Fredholm section in a suitable Hilbert bundle over $\mathcal{B}^*$ (cf. [5, section 4.2]). Indeed, according to [5, remark 4.3], the first component of any solution to the (J, τ_0)-vortex equations is irreducible, that is belongs to $\mathcal{B}^*$.

Since $\mathcal{J}_0$ is path connected (it is actually contractible), we can apply the homotopy invariance of the virtual fundamental class in the sense of Brussee (cf. [5, theorem 6.4]) to a path $[0, 1] \to \mathcal{J}_0$ joining some arbitrary point $J \in \mathcal{J}_0$ to J_0, and deduce that the Hamiltonian Gromov-Witten invariant is independent of the choice of the almost complex structure $J \in \mathcal{J}_0$. $\square$

(iii) The key argument used in the previous proof is the homotopy invariance of the virtual fundamental class à la Brussee. This should allow to relate the gauge theoretical and the Hamiltonian invariants in much wider contexts.

We wish to clarify now the enumerative meaning of the Hamiltonian invariants. We say that a character $\eta \in \mathcal{X}^*(T) \cong A^1_T$ is nef if the corresponding line bundle $\mathcal{O}_X(\eta) := \Omega \times_\eta \mathbb{C} \to X$ is nef; on toric varieties, nef line bundles are globally generated, and the nef characters generate the closure of the Kähler cone.

Proposition 5.2. *Let C be a smooth and irreducible curve of genus g, and $\underline{d} = (d_\rho)_\rho$ a multi-degree such that $d_\rho > 2g-1$ for all ρ. We consider distinct points $\zeta_1, \dots, \zeta_m$ on C, with $m := |\underline{d}| - n(g-1)$, and the not necessarily distinct nef characters $\eta_1, \dots, \eta_m \in \mathcal{X}^*(T)$. For each $i = 1, \dots, m$ we take a general divisor $Y_i \hookrightarrow X$ in the linear system defined by η_i. Then*

$$I^g_{\underline{d}}(\eta_1 \cdot \dots \cdot \eta_m) = \#\{u \in \mathsf{Mor}_{\underline{d}}(C, X) \mid u(\zeta_i) \in Y_i,\ \forall i = 1, \dots, m\}.$$

Proof. For $i = 1, \dots, m$ we take the restriction $\mathcal{P}_{\underline{\delta},\zeta_i} \to \mathcal{J}^l$ at the marked points, and we consider the fibre product

$$\Pi_{\underline{\zeta}} := \big((\mathcal{P}_{\underline{\delta},\zeta_1} \times_{\mathcal{J}^l} \dots \times_{\mathcal{J}^l} \mathcal{P}_{\underline{\delta},\zeta_m}) \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T \xrightarrow{T^m} V_{\underline{d}}(C).$$

The morphism Φ induces the evaluation $\Phi_{\underline{\zeta}} : \Pi_{\underline{\zeta}} \to (\mathbb{C}^r)^m$, and we notice that it is T^m-equivariant.

Let $\eta \in \mathcal{X}^*(T)$ be a nef character: using the notations in proposition A.1, $H^0(X, \mathcal{O}_X(\eta)) = S_\eta$, and we have a T-equivariant morphism

$$F_\eta : \mathbb{C}^m \to S_\eta^\vee, \quad z \longmapsto l_z : f \mapsto f(z),$$

with T acting by η on $S_\eta^\vee$. Since η is globally generated, the morphism $\hat{F}_\eta : X \to \mathbb{P}(S_\eta^\vee)$ is everywhere defined, and therefore $F_\eta(\Omega) \subseteq S_\eta^\vee \setminus \{0\}$.

The composition

$$\Pi_{\underline{\zeta}} \xrightarrow{\Phi_{\underline{\zeta}}} (\mathbb{C}^r)^m \xrightarrow{F=(F_{\eta_i})_i} \bigtimes_{i=1}^m S_{\eta_i}^\vee$$

induces the section

$$\phi = (\phi_i)_i : V_{\underline{d}}(C) \longrightarrow \Pi_{\underline{\zeta}} \times_{T^m} \left(\times_i S_{\eta_i}^\vee\right) =: \oplus_i \mathcal{E}_i = \mathcal{E},$$

where the $\mathcal{E}_i \to V_{\underline{d}}(C)$ are vector bundles. We blow the $\mathcal{E}_i$'s up and obtain

$$\tilde{\mathcal{E}} := \oplus_i \operatorname{Bl}_0 \mathcal{E}_i = \Pi_{\underline{\zeta}} \times_{T^m} \left(\times_i \operatorname{Bl}_0(S_{\eta_i}^\vee)\right) \longrightarrow \bigtimes_{i=1}^m \mathbb{P}(S_{\eta_i}^\vee).$$

We consider the diagram

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[dr] & \phi^*\tilde{\mathcal{E}} \arrow[r] \arrow[d] & \tilde{\mathcal{E}} \arrow[d] \\ & V_{\underline{d}}(C) \arrow[r, "\phi"] & \mathcal{E} \end{tikzcd}$$

where V' is the component of $\phi^*\tilde{\mathcal{E}}$ which contains $\mathsf{Mor}_{\underline{d}}(C, X)$; this statement makes sense since $\phi_i\big(\mathsf{Mor}_{\underline{d}}(C, X)\big) \subseteq \mathcal{E}_i \setminus \{0\} \subset \mathrm{Bl}_0\mathcal{E}_i$ for all i. Let $(V')^o \subset V'$ be the open subset which lies over $\mathsf{Mor}_{\underline{d}}(C, X)$: then $(V')^o \to \mathsf{Mor}_{\underline{d}}(C, X)$ is an isomorphism.

For $i = 1, \dots, m$, let $f_i \in S_{\eta_i}$ be the equation defining the divisor $Y_i \hookrightarrow X$; f_i defines a hyperplane $\mathcal{H}_i \hookrightarrow \mathbb{P}(S^\vee_{\eta_i})$. Since $m = \dim V'$ and the Y_i's are in general position, the intersection

$$\text{Image of } \big(V' \setminus (V')^o\big) \cap \bigtimes_{i=1}^{m} \mathcal{H}_i = \emptyset.$$

Actually, the assumption that the Y_i's are general means precisely that the hyperplanes $\mathcal{H}_i$ are transverse to V', and their intersection is disjoint from $V' \setminus (V')^o$.

The composition $\Pi_{\underline{\zeta}} \xrightarrow{\Phi_{\underline{\zeta}}} (\mathbb{C}^r)^m \xrightarrow{f=(f_i)_i} \mathbb{C}^m$ defines the section

$$\hat{f} : V_{\underline{d}}(C) \to \Pi_{\underline{\zeta}} \times_{T^m} \mathbb{C}^m = \oplus_i \Lambda_{\eta_i}.$$

Our discussion implies that its zero set is disjoint from $V_{\underline{d}}(C) \setminus \mathsf{Mor}_{\underline{d}}(C, X)$, and therefore

$$Zero(\hat{f}) = \{u \in \mathsf{Mor}_{\underline{d}}(C, X) \mid u(\zeta_i) \in Y_i,\ \forall i = 1, \dots, m\}.$$

On the other hand, $I^g_{\underline{d}}(\eta_1 \cdot \ldots \cdot \eta_m)$ equals the number of zeros of $\hat{f}$, and this proves our proposition. □

6. Localization

In this section we will apply the localization method developed in [10] for computing intersection products of the Λ_ρ's on $W_{\underline{d}}$ with respect to the action of the torus S, the ultimate goal being to compute the Hamiltonian invariants of X. We are going to localize in the smooth case, that is we consider multi-degrees $\underline{d}$ such that $d_\rho > 2g - 1$ for all ρ. However, the final formulae will hold for arbitrary multi-degrees due to the recursion formula (5.3).

First of all we have to make the S-action on $W_{\underline{d}}$ explicit, and to describe the corresponding fixed point set $W^S_{\underline{d}}$. Since Z_W defined by (3.4) is $(\mathbb{C}^*)^r$-invariant, its complement $\mathcal{W}^o$ is still $(\mathbb{C}^*)^r$-invariant, so that $S = (\mathbb{C}^*)^r/T$ acts on $W_{\underline{d}} = \mathcal{W}^o/T$ and moreover the evaluation map (5.1) is S-equivariant. This implies that if $[\underline{s}] \in W^S_{\underline{d}}$, then for all $\zeta \in C$ such that $\underline{s}(\zeta) \notin Z_W$, $ev_{[\underline{s}]}(\zeta) \in X^S$. But X^S consists of finitely many points: they correspond in a bijective fashion to the n-dimensional cones of Σ and their number equals the Euler characteristic of X. For $x \in X^S$, we shall denote $\sigma(x)$ the corresponding n-dimensional cone of Σ, and by $\bar{O}_x \subset \mathbb{C}^r$ the closure of the T-orbit above x. In fact $\bar{O}_x$ is the linear l-dimensional subspace of $\mathbb{C}^r$ defined by the equations

$$\bar{O}_x = \{z^\rho = 0 \mid \rho \in \sigma(x)\}.$$

Our discussion implies that for any $[\underline{s}] \in W_{\underline{d}}^S$, the image of the evaluation $ev_{[\underline{s}]}$ is a point $x \in X^S$, and this in turn means that $\underline{s} \in \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)$. What we have obtained so far is that

$$W_{\underline{d}}^S \subset \bigcup_{x \in X^S} \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)^o / T =: \bigcup_{x \in X^S} W_{\underline{d}}(x),$$

and our goal is to show that this inclusion is in fact an equality.

We are going to check that the component $\Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0})^o / T$ is fixed by S, for $x_0 \in X^S$ the point corresponding to the cone $\sigma(x_0) = \langle e_{l+1}, \dots, e_r \rangle$ of Σ. With respect to this choice of coordinates, the T-action on $\mathbb{C}^r$ is given by (2.7) and

$$[t_1, \dots, t_l, t_{l+1}, \dots, t_r] = [1, \dots, 1, \chi_1^{-1}(t')t_{l+1}, \dots \chi_l^{-1}(t')t_r] \text{ in } S,$$

for $t' = (t_1, \dots, t_l)$. Now is clear that any point $[s] \in W_{\underline{d}}(x_0)$ is fixed because $\bar{O}_{x_0} = \{z^\rho = 0 \mid \rho = l+1, \dots, n\}$, so that $W_{\underline{d}}(x_0) \subset W_{\underline{d}}^S$. But we could have described the action of T on $\mathbb{C}^r$ using the coordinates furnished by any other n-dimensional cone of Σ and the conclusion would have been the same.

Proposition 6.1. *The fixed point set of the S-action on $W_{\underline{d}}$ is*

$$W_{\underline{d}}^S = \bigcup_{x \in X^S} W_{\underline{d}}(x), \ \textit{with } W_{\underline{d}}(x) := \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_x)^o / T.$$

Moreover, for any $x \in X^S$, $W_{\underline{d}}(x)$ is defined by the fibre product

$$\begin{array}{ccc} W_{\underline{d}}(x) & \longrightarrow & \prod_{\rho \notin \sigma(x)} \mathbb{P}(\mathcal{W}_\rho) \\ \downarrow & & \downarrow \\ \mathcal{J}^r & \xrightarrow{\mathrm{pr}_x} & \prod_{\rho \notin \sigma(x)} \mathcal{J}. \end{array}$$

Proof. We must prove the second claim: we are going to check it for $x_0 \in X^S$ only, the general statement coming from the symmetry of the problem. We observe that $Z_X \cap \bar{O}_{x_0} = \bar{O}_{x_0} \setminus O_{x_0} = \bigcup_\lambda (\{z^\lambda = 0\} \cap \bar{O}_{x_0})$, because on O_{x_0} the torus T acts freely. Therefore

$$\begin{aligned} \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0})^o &= \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} \bar{O}_{x_0}) \setminus \Gamma(C, \mathcal{P}_{\underline{d}} \times_{(\mathbb{C}^*)^r} (Z_X \cap \bar{O}_{x_0})) \\ &= \prod_{\lambda=1}^l \big(\mathcal{W}_\lambda \setminus \{0\}\big), \end{aligned}$$

and the statement follows because $T \cong (\mathbb{C}^*)^l$ acts componentwise. □

The moral is that no matter what $W_{\underline{d}}$ looks like, its fixed point set for the torus action has a very down-to-earth description. A first by-product is an explicit formula for the Euler number of the fibre X' of $W_{\underline{d}} \to \mathcal{J}^r$.

Corollary 6.1.

$$\chi(X') = \sum_{x \in X^S} \prod_{\rho \notin \sigma(x)} N_\rho.$$

Proof. Since X' is a toric variety, its Euler characteristic coincides with the number of fixed points under the $(\mathbb{C}^*)^R/T$-action. The big torus of X' contains S, and therefore the fixed point set is contained in the union of the S-fixed subvarieties. But these are just products of projective spaces on which $(\mathbb{C}^*)^R$ acts in the standard fashion. □

For applying the localization formula we must know the action of S on the normal bundles to the fixed subvarieties.

Lemma 6.1. *For any $x \in X^S$, the normal bundle of the fixed component $W_{\underline{d}}(x)$ of $W_{\underline{d}}$ fits in the following diagram:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(\mathcal{L}ie\,T) \arrow[r] \arrow[d, equal] & \bigoplus_{\rho\notin\sigma(x)} q^*\mathcal{W}_\rho \otimes \Lambda_\rho \arrow[r] \arrow[d] & \mathsf{T}_{W_{\underline{d}}(x)/\mathfrak{J}^r} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(\mathcal{L}ie\,T) \arrow[r] & \bigoplus_{\rho=1}^{r} q^*\mathcal{W}_\rho \otimes \Lambda_\rho \arrow[r] \arrow[d] & \mathsf{T}_{W_{\underline{d}}/\mathfrak{J}^r} \arrow[r] \arrow[d] & 0 \\
 & & \bigoplus_{\rho\in\sigma(x)} q^*\mathcal{W}_\rho \otimes \Lambda_\rho \arrow[r, "\cong"] \arrow[d] & \mathsf{N}_x := \mathsf{N}_{W_{\underline{d}}(x)|W_{\underline{d}}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{6.1}$$

For the trivial action of S on $\mathcal{W}_\rho$ and for the action

$$\begin{aligned} &S \times \Lambda_\rho \longrightarrow \Lambda_\rho \text{ given by} \\ &[t] \times [\underline{s}, a] := [t \times \underline{s}, t_\rho a], \quad \forall\, [t] \in S \text{ and } [\underline{s}, a] \in \Lambda_\rho, \end{aligned} \tag{6.2}$$

all the homomorphisms in the diagram above are S-equivariant.

Proof. Since $\mathcal{W}^o \to W_{\underline{d}}$ is a principal T-bundle, the following exact sequence on $W_{\underline{d}}$ is S-equivariant

$$0 \longrightarrow \mathcal{O}(\mathcal{L}ie\,T) \longrightarrow \mathsf{T}^{\mathrm{inv}}_{\mathcal{W}^o/\mathfrak{J}^r} \longrightarrow \mathsf{T}_{W_{\underline{d}}/\mathfrak{J}^r} \longrightarrow 0, \tag{6.3}$$

where $\mathsf{T}^{\mathrm{inv}}_{\mathcal{W}^o/\mathfrak{J}^r}$ denotes the S-invariant relative tangent bundle to the total space of $\mathcal{W}^o$. But $\mathcal{W}^o$ is an open subset in a vector bundle over $\mathfrak{J}^r$, so that the relative tangent bundle is canonically isomorphic to $Q^*\mathcal{W} = \oplus_\rho Q^*\mathcal{W}_\rho$, for $Q : \mathcal{W} \to \mathfrak{J}^r$ the projection. As T preserves the decomposition of $\mathcal{W}$,

$$\mathsf{T}^{\mathrm{inv}}_{\mathcal{W}^o/\mathcal{J}^r} \cong Q^*\mathcal{W}/T = \bigoplus_{\rho=1}^{r} Q^*\mathcal{W}_\rho/T.$$

We observe now that $Q^*\mathcal{W}_\rho/T \cong q^*\mathcal{W}_\rho \otimes \Lambda_\rho$, the isomorphism being given by

$$[\underline{s}, w_\rho] \longmapsto w_\rho \otimes [\underline{s}, 1]. \tag{6.4}$$

This proves the exactness of the middle row in the diagram (6.1). A similar argument proves the exactness of the first horizontal sequence, and the last row is now a simple consequence.

The very important thing which must be clarified yet is the way how S acts on $q^*\mathcal{W}_\rho \otimes \Lambda_\rho$. The sequence (6.3) being S-equivariant, we have to describe the induced action on $q^*\mathcal{W}_\rho \otimes \Lambda_\rho$ under the isomorphism (6.4). For $[t] \in S$,

$$\begin{aligned}[t] \times \big(w_\rho \otimes [\underline{s}, 1]\big) &= [t] \times [\underline{s}, w_\rho] = [t \times \underline{s}, t_\rho w_\rho] \mapsto t_\rho w_\rho \otimes [t \times \underline{s}, 1] \\ &= w_\rho \otimes [t \times \underline{s}, t_\rho],\end{aligned}$$

so that we can see that indeed the S-action on $\mathcal{W}_\rho$ is trivial while the action on Λ_ρ is given by (6.2). □

The next step is the computation of the equivariant first Chern classes for the restrictions of Λ_ρ to the fixed components $W_{\underline{d}}(x)$. Before proceeding we notice that since $S = M^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$, there is a natural ring isomorphism $A^*_S \cong \mathrm{Sym}^\bullet M$, where A^*_S denotes the S-equivariant Chow group of a point.

Lemma 6.2. *For $x \in X^S$, denote $(u_\rho(x))_{\rho\in\sigma(x)} \subset M$ the dual basis to $(e^\rho)_{\rho\in\sigma(x)} \subset M^\vee$ formed by the integral generators of $\sigma(x)$. Then*

$$\begin{aligned} c_1^S(\Lambda_\rho|_{W_{\underline{d}}(x)}) &= \Lambda_\rho|_{W_{\underline{d}}(x)} + u_\rho(x), \quad \forall \rho \in \sigma(x), \\ c_1^S(\Lambda_\rho|_{W_{\underline{d}}(x)}) &= \Lambda_\rho|_{W_{\underline{d}}(x)}, \quad \forall \rho \notin \sigma(x), \end{aligned}$$

and the equivariant Euler characteristic of the normal bundle $\mathsf{N}_x \to W_{\underline{d}}(x)$ is

$$\tilde{e}(\mathsf{N}_x) = \prod_{\rho\in\sigma(x)} (u_\rho(x) + \Lambda_\rho)^{N_\rho} \cdot \exp\Big(-\frac{\theta_\rho}{u_\rho(x) + \Lambda_\rho}\Big).$$

Proof. It is clear that $c_1^S(\Lambda_\rho|_{W_{\underline{d}}(x)}) = \Lambda_\rho|_{W_{\underline{d}}(x)} + u$, for some $u \in A^*_S$, and this element is precisely the weight of the action of S on the stalk $\Lambda_\rho|_{[\underline{s}]}$ at some point $[\underline{s}] \in W_{\underline{d}}(x)$. Again, we shall make the computations for x_0 only: in this case the assignment $(\tau_{l+1}, \dots, \tau_r) \mapsto [1, \dots, 1, \tau_{l+1}, \dots, \tau_r]$ defines an isomorphism $(\mathbb{C}^*)^n \xrightarrow{\cong} S$, and is easy to see that

$$(\mathbb{C}^*)^n \text{ acts on } \Lambda_\rho \begin{cases} \text{trivially, for } \rho = 1, \dots, l \textit{ i.e. } \rho \notin \sigma(x_0), \\ \text{by } \tau_\rho, \text{ for } \rho = l+1, \dots, r \textit{ i.e. } \rho \in \sigma(x_0). \end{cases}$$

A short computation shows that the isomorphism above is induced precisely by the choice of the dual basis to $(e^{l+1}, \dots, e^r)$, and the conclusion follows.

The formula for the normal bundle is a direct consequence of the lemma 6.1 and of the fact that the total Chern class of the Picard bundle $\mathcal{W}_\rho \to \mathfrak{J}$ is $c(\mathcal{W}_\rho) = \exp(-\theta_\rho)$, with θ_ρ the class of the theta divisor (the lower index ρ indicates that we are on the ρ^{th} copy of $\mathfrak{J}$ in $\mathfrak{J}^r$). □

As we have already said, we are going to apply the localization formula for computing the Hamiltonian invariants of the toric variety X: for positive integers $m_1, \dots, m_r$, we wish to compute the push-forward $q_*\big((\Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r}) \cap [V_{\underline{d}}(C)]\big) \in A_*(\mathfrak{J}^r)$. For shorthand, we are going to write $\phi := \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r}$, and $\tilde{\phi}$ for the equivariant analog of ϕ.

Theorem 6.1. *The Poincaré dual of $q_*\big([V_{\underline{d}}(C)] \cap \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r}\big)$ is the constant term of the polynomial*

$$\sum_{x \in X^S} (q_x)_* \left[\prod_{\rho \notin \sigma(x)} \Lambda_\rho^{m_\rho} \cdot \prod_{\rho \in \sigma(x)} (u_\rho(x) + \Lambda_\rho)^{m_\rho - N_\rho} \cdot \exp\left(\frac{\theta_\rho}{u_\rho(x) + \Lambda_\rho} \right) \right].$$

Proof. We recall from [10, theorem 2] that if $\iota_x : W_{\underline{d}}(x) \hookrightarrow W_{\underline{d}}$ denotes the inclusion,

$$\tilde{\phi} = \sum_{x \in X^S} (\iota_x)_* \frac{\iota_x^* \tilde{\phi}}{\tilde{e}(\mathsf{N}_x)}.$$

Since S acts trivially on $\mathfrak{J}^r$, by taking the proper push-forward under $q : W_{\underline{d}} \to \mathfrak{J}^r$, we obtain

$$q_* \tilde{\phi} = \sum_{x \in X^S} (q_x)_* \frac{\iota_x^* \tilde{\phi}}{\tilde{e}(\mathsf{N}_x)} \in A_S^*(\mathfrak{J}^r) = A_S^* \otimes A^*(\mathfrak{J}^r),$$

and our conclusion follows now from lemma 6.2. □

The computation of this constant term is rather difficult because in most cases the constant term of the sum is not equal to the sum of the individual constant terms. For concrete computations, it is useful to localize with respect to a generic 1-parameter subgroup $\mu : \mathbb{C}^* \to S$; it can be canonically identified with an element $\mu \in N$. In our case, by generic we mean that the fixed point set X^μ for the induced $\mathbb{C}^*$-action on X coincides with X^S.

Corollary 6.2. *Consider an element $\mu \in N$ which defines a generic 1-parameter subgroup of S, and denote $\mu_\rho(x) := \langle \mu, u_\rho(x) \rangle$ for all $x \in X^\mu = X^S$ and $\rho = 1, \dots, r$. Then $q_*\phi$ equals the constant term of*

$$\sum_{x \in X^S} (q_x)_* \left[\prod_{\rho \notin \sigma(x)} \Lambda_\rho^{m_\rho} \cdot \prod_{\rho \in \sigma(x)} (\mu_\rho(x) u + \Lambda_\rho)^{m_\rho - N_\rho} \cdot \exp\left(\frac{\theta_\rho}{\mu_\rho(x) u + \Lambda_\rho} \right) \right]$$

viewed as an element in $A^*(\mathfrak{J}^r)[u]$.

The advantage of this writing is that, since we are dealing with polynomials in only one variable, the constant term of the sum equals the sum of the individual constant terms. Computing these is straightforward but requires a little patience.

Lemma 6.3. *For $p \geq 0$,*

$$(u+\Lambda)^p \exp\Big(\frac{\theta}{u+\Lambda}\Big) = \sum_{0\leq k\leq p}\Bigg[\sum_{0\leq a\leq k}\binom{p-a}{k-a}\Lambda^{k-a}\frac{\theta^a}{a!}\Bigg]u^{p-k}$$
$$+\theta^{p+1}\sum_{0\leq k}\Bigg[\sum_{0\leq b\leq k}(-1)^{k-b}\binom{k}{k-b}\Lambda^{k-b}\frac{\theta^b}{(p+1+b)!}\Bigg]\frac{1}{u^{k+1}},$$

while for $p \geq 1$,

$$\frac{1}{(u+\Lambda)^p}\exp\Big(\frac{\theta}{u+\Lambda}\Big) = \sum_{0\leq k}\Bigg[\sum_{0\leq b\leq k}(-1)^{k-b}\binom{p+k-1}{k-b}\Lambda^{k-b}\frac{\theta^b}{b!}\Bigg]\frac{1}{u^{p+k}}.$$

Using the equalities

$$(q_x)_*\Big(\prod_{\rho\notin\sigma(x)}\Lambda_\rho^{k_\rho}\Big) = \begin{cases}\prod_{\rho\notin\sigma(x)}\frac{\theta_\rho^{k_\rho-N_\rho+1}}{(k_\rho-N_\rho+1)!} & \text{if } N_\rho-1\leq k_\rho\leq N_\rho+g-1\ \forall\rho,\\ 0 & \text{otherwise}\end{cases} \tag{6.5}$$

it is possible, but not so easily, to compute the explicit value of our invariants.

Let us turn our attention to the case of the invariants of maximal degree: these ones are $\mathbb{Z}$-valued, and are related to the enumerative geometry of X. In the formula 6.2, we see that the contribution to the invariant of the term corresponding to $x \in X^S$ equals

$$\Big\langle (q_x)_*\Big[\text{coeff}(x)\cdot\textstyle\prod_{\rho\notin\sigma(x)}\Lambda_\rho^{d_\rho}\Big], [G_C]\Big\rangle = \Big\langle \text{coeff}(x)\cdot\textstyle\prod_{\rho\notin\sigma(x)}\frac{\theta_\rho^g}{g!}, \big[\psi(\textstyle\prod_{\rho\notin\sigma(x)}\mathcal{J})\big]\Big\rangle$$
$$= \text{coeff}(x),$$

the coefficient of $\prod_{\rho\notin\sigma(x)}\Lambda_\rho^{d_\rho}$.

Example 6.1. A straightforward application of the localization formula is that

$$I^1_{\underline{0}}(1) = \chi(X).$$

Indeed, the recursive relations (5.3) imply that $I^1_{\underline{0}}(1) = I^1_{\underline{d}}(\chi_1^{d_1}\cdot\ldots\cdot\chi_r^{d_r})$ for any multi-degree $\underline{d} = (d_\rho)_\rho$ obeying the linear relations (2.9); we choose $\underline{d}$ such that $d_\rho > 1$ for all ρ. The Jacobian of an elliptic curve E is isomorphic to the curve itself, and the localization formula gives

$$I^1_{\underline{d}}(\chi_1^{d_1}\cdot\ldots\cdot\chi_r^{d_r}) = \Big\langle \sum_{x\in X^S}\prod_{\lambda\notin\sigma(x)}\theta_\lambda\,,\,[G_E]\Big\rangle = \#X^S = \chi(X).$$

This computation illustrates the importance of the virtual fundamental class too. In this case, $V_{\underline{0}}(E) \cong X$ and $[\![V_{\underline{0}}(E)]\!] \in A_0(X)$ is represented by a zero-dimensional subscheme of length $\chi(X)$.

We conclude this section with a vanishing result, which is a direct consequence of theorem 6.1.

Proposition 6.2. *If* $\pi \subset \Sigma(1)$ *is a primitive collection, and* $\{m_\rho\}_\rho$ *are positive integers such that* $m_\rho \geq d_\rho + 1$ *for all* $\rho \in \pi$*, then* $I^g_{\underline{d}}(m_1, \ldots, m_r) = 0$*. In other words,*

$$I^g_{\underline{d}}(m_1, \ldots, m_r) \neq 0 \implies \exists \tau \in \Sigma \text{ s.t. } \left\{\rho \mid m_\rho - d_\rho \geq 1\right\} = \tau(1).$$

In this latter case, only those $x \in X^S$ *contribute to the invariant which have the property that* $\sigma(x) \supseteq \tau$.

Proof. Indeed, for any $x \in X^S$, there exists $\rho \in \pi \setminus \sigma(x)$. Since $m_\rho \geq d_\rho + 1$, it follows $m_\rho - N_\rho \geq g$ and we conclude using the relations (6.5). □

We observe that there are toric varieties which do not possess primitive collections π, and integers $\{m_\rho\}_\rho$ with the property above: $e.g.$ the projective spaces.

7. Degenerations: Part I

In the subsequent sections we will exclusively study the invariants of highest degree. Though theoretically the formulae found in the previous section compute all our Hamiltonian invariants (with the help of a good computer!), they give no insight into their structure. In proposition 5.1 we have proved that these invariants do not depend on the conformal structure of C, so we may allow it to degenerate. There are two kind of interesting degenerations in the Gromov-Witten theories: the first one is the degeneration of a smooth curve to an irreducible curve with one node, and the second one is the degeneration to a reducible curve with two smooth and irreducible components, which meet at one ordinary double point. These usually induce recursive formulae for the invariants. In this section we will treat the first degeneration type, which is easier to deal with. We will find that this issue is related to the problem of compactifying the relative Jacobian; for the convenience of the reader we have recalled some standard facts about it in appendix B.

Let us consider a flat family $\mathscr{C} \to \triangle$ of reduced and irreducible curves of genus g over a 1-dimensional base, having the property that all the fibres are smooth, except the special fibre C_o over $o \in \triangle$, which is assumed to have exactly one node. We will denote by $\tilde{C}_o$ its normalisation, with $n : \tilde{C}_o \to C_o$ the normalisation map, and by $z', z'' \in \tilde{C}_o$ the two points which are identified by n. Possibly after shrinking $\triangle$ and after étale base change, we may assume that there is a section $s : \triangle \to \mathscr{C}$ such that $s(o) \in C_o$ is a smooth point. According to theorem B.2, the compactified Jacobian $\mathscr{J} \to \triangle$ exists, together with the universal Poincaré sheaf $\mathscr{L}_0 \to \mathscr{J} \times_\triangle \mathscr{C}$ which is trivialised along the divisor $\mathscr{J} \times_\triangle s(\triangle)$; for any integer d, we denote $\mathscr{L}_d$ the Poincaré sheaf of degree d.

We choose and fix for the rest of this section a multi-degree $\underline{d} = (d_\rho)_\rho$ which obeys the linear relations (2.9). Unless otherwise stated, we will further assume that $d_\rho \geq \max\left\{\frac{n}{2} + g - 1, 2g + 1\right\}$ for all ρ. For such a choice, the moduli spaces

$V_{\underline{d}}(C_t)$, $t \in \triangle \setminus \{o\}$, are smooth and have the expected dimension, and the Chow ring of the fibre X' of $V_{\underline{d}}(C_t) \to \mathfrak{J}^l$ approximates well A^*_T (see remark 4.1).

For $t \in \triangle \setminus \{o\}$, the morphisms $\psi_t : \mathfrak{J}^l_t \to \mathfrak{J}^r_t$ defined in (2.12) fit into a family of morphisms over $\triangle \setminus \{o\}$. However, this family *does not extend* to a morphism over the whole $\triangle$ since the compactified Jacobian $\mathfrak{J}_o$ is not an abelian variety. Nevertheless $\psi_o : \mathsf{Pic}^0(C_o)^l \to \mathsf{Pic}^0(C_o)^r$ is well-defined and we get a rational map

$$\psi : \mathscr{J}^l \dashrightarrow \mathscr{J}^r.$$

We define $\mathcal{G}_o$ to be the limit of the subvarieties $\psi_t(\mathfrak{J}^l_t) \hookrightarrow \mathfrak{J}^r_t$, in a suitable relative Hilbert scheme of $\mathscr{J}^r/\triangle$; this limit is well-defined by the relative criterion of properness. In general, $\mathcal{G}_o$ is reducible and non-reduced, and it seems rather difficult to describe it accurately. In fact, the most part of this section is devoted to investigate its structure. What we can immediately say however is that one of the irreducible components of $\mathcal{G}_o$ is the closure of the image of $\psi_o : \mathfrak{J}^l_o \dashrightarrow \mathfrak{J}^r_o$, which we denote by G_o; this component comes with multiplicity one, because ψ_o is defined on $\mathsf{Pic}^0(C_o)$. Then $\mathcal{G}_o = G_o \cup G^{\mathrm{bad}}$, where G^{bad} stays for the rest of the components of $\mathcal{G}_o$. We consider now the diagram (B.3)

$$\begin{tikzcd} & \mathbb{J}^r_o \arrow[dl, "\nu"'] \arrow[dr, "\mathrm{pr}"] & \\ \mathfrak{J}^r_o \arrow[rr, dashed] & & \tilde{\mathfrak{J}}^r_o \end{tikzcd}$$

and denote $\tilde{G}_o := \tilde{\psi}(\tilde{\mathfrak{J}}^l_o)$, where $\tilde{\mathfrak{J}}_o$ is the Jacobian of $\tilde{C}_o$ and $\tilde{\psi}$ is given by (2.12). We define further

$$\mathbb{G}_o := \mathcal{G}_o \times_{\mathfrak{J}^r_o} (\mathbb{J}_o)^r = \nu^{-1}\mathcal{G}_o \hookrightarrow (\mathbb{J}_o)^r.$$

Lemma 7.1. (i) *$\mathcal{G}_o$ is pure dimensional, of dimension lg.*

(ii) *Each irreducible component of $\mathbb{G}_o$ projects by* pr *onto $\tilde{G}_o$.*

(iii) *$G_o \hookrightarrow \mathfrak{J}^r_o$ is irreducible and reduced.*

(iv) *The map $\nu^{-1}G_o \to \tilde{G}_o$ is a locally trivial fibration whose fibres are l-dimensional projective toric varieties.*

Proof. (i) The statement follows from the very definition of $\mathcal{G}_o$.

(ii) Consider a point $j \in \mathcal{G}_o$: then we find a sequence $t_n \to o$, and points $j_n \in \psi_{t_n}(\mathfrak{J}^l_{t_n})$ which converge to j. For each n we find some morphism $v_n : C_{t_n} \to X$ such that $j_n = [v_n^*\Omega \to C_{t_n}]$. The Gromov compactness theorem implies that, after passing to a subsequence, the morphisms v_n converge to a stable map $v' : C'_o \to X$, where C'_o is obtained by attaching a finite number of trees of $\mathbb{P}^1$'s at smooth points of C_o and/or inserting such a tree at its double point. Then $\mathrm{pr}(j) \in \tilde{\mathfrak{J}}^r_o$ represents the holomorphic type of the principal bundle $(v'|_{\tilde{C}_o})^*\Omega \to \tilde{C}_o$, and therefore belongs to $\tilde{G}_o$. This proves that $\mathrm{pr}(\mathbb{G}_o) \subseteq \tilde{G}_o$.

We notice first that $\mathsf{Pic}^0(C_o)^l$ naturally acts on $\mathcal{G}_o$, and therefore on $\mathbb{G}_o$ too. Let us fix now an irreducible component $\mathbb{K} \hookrightarrow \mathbb{G}_o$; this one is invariant under the action, and moreover the projection $\mathbb{K} \to \tilde{G}_o$ is equivariant for the $\mathsf{Pic}^0(C_o)^l$-action on $\mathbb{K}$ and the $\mathsf{Pic}^0(\tilde{C}_o)^l$-action on $\tilde{G}_o$. Since the latter one is transitive, we deduce that $\mathbb{K} \to \tilde{G}_o$ is surjective.

(iii) This statement is clear since G_o is simply the closure of $\psi_o\big(\mathsf{Pic}^0(C_o)^l\big) \subset \mathfrak{J}_o^r$.

(iv) We notice that $\nu^{-1}G_o$ is irreducible, and we know from (i) that in this case the morphism $\nu^{-1}G_o \to \tilde{G}_o$ is surjective and equivariant with respect to the $\mathsf{Pic}^0(C_o)^l$-action on G_o and the $\mathsf{Pic}^0(\tilde{C}_o)^l$-action on $\tilde{G}_o$. Since the kernel of $\mathsf{Pic}^0(C_o)^l \to \mathsf{Pic}^0(\tilde{C}_o)^l$ is isomorphic to T, and this one *operates effectively* on $\nu^{-1}G_o$, we deduce that all the fibres of $\nu^{-1}G_o \to \tilde{G}_o$ are isomorphic to the projective variety obtained by compactifying $T \subset (\mathbb{C}^*)^r \subset (\mathbb{P}^1)^r$. The local triviality of $\nu^{-1}G_o \longrightarrow \tilde{G}_o$ is implied by the same equivariance property.

□

In order to deduce recursive relations for our invariants, we must first construct the degeneration of the compactification $V(C_t)$ constructed in corollary 3.1 as $t \to o \in \triangle$. It turns out that this is a subtle issue, since in the $t \to o$ limit may appear some 'unexpected components' which project onto $G^{\text{bad}} \hookrightarrow \mathcal{G}_o$.

We consider again the Picard sheaves $\mathscr{W}_\rho := p_*\mathscr{L}_{d_\rho} \to \mathscr{J}$, and let as usual $\mathscr{W} := \oplus_\rho \mathscr{W}_\rho$, $\mathscr{W}^o := \mathscr{W} \setminus \mathscr{Z}_W$; the assumption on $\underline{d}$ implies that they are locally free for all ρ. Define now $\mathscr{V} := \mathscr{W}|_{\overline{\text{Image}(\psi)}}$, and consider the 'nice' open subset $\mathscr{V}^o := \mathscr{V} \cap \mathscr{W}^o$.

Proposition 7.1. (i) *The quotient $\mathscr{V}^o/T$ exists and it is projective over $\triangle$; its fibre over $t \in \triangle \setminus \{o\}$ coincides with the compactification $V_{\underline{d}}(C_t)$ constructed in corollary 3.1. One of the irreducible components of the central fibre $\mathcal{V}_{\underline{d}}(C_o)$ is the compactification $V_{\underline{d}}(C_o)$ of the space of morphisms $C_o \to X$ having multi-degree $\underline{d}$.*

(ii) *$\mathcal{V}_{\underline{d}}(C_o)$ is the limit of the subvarieties $V_{\underline{d}}(C_t) \hookrightarrow \mathscr{W}_t^o/T$ in a suitable relative Hilbert scheme of $\mathscr{W}^o/T \to \triangle$.*

(iii) *The line bundles $\Lambda_{\rho,t} \to V_{\underline{d}}(C_t)$ fit into the family $\boldsymbol{\Lambda}_\rho := \mathscr{V}^o \times_{\chi_\rho} \mathbb{C} \to \mathscr{V}^o/T$.*

(iv) *There is a rational map $N : V_{\underline{d}}(C_o) \dashrightarrow V_{\underline{d}}(\tilde{C}_o)$ which is birational on its image, and the pull-back under N of $\tilde{\Lambda}_\rho \to V_{\underline{d}}(\tilde{C}_o)$ coincide with $\Lambda_{\rho,o} \to V_{\underline{d}}(C_o)$.*

Proof. (i) The quotient $\mathscr{V}^o/T$ exists since fibrewise we are in the situation of [15, lemma 5.2]; in fact $\mathscr{V}^o \to \mathscr{V}^o/T$ is a principal T-bundle. However, we wish to stress that the geometric fibre $\mathcal{V}_{\underline{d}}(C_o)$ is in general reducible and even non-reduced; all we are able to say is that $\mathcal{V}_{\underline{d}}(C_o) \to \mathcal{G}_o$ is a locally trivial toric fibre bundle. It contains the 'nice' component $V_{\underline{d}}(C_o)$, which is by definition the compactification of the space of morphism $C_o \to X$ of multi-degree $\underline{d}$; this one projects onto $G_o \hookrightarrow \mathcal{G}_o$.

(ii) On one hand, we know that $\mathcal{G}_o = \lim_{t\to o} \psi_t(\mathfrak{J}_t^l)$, and, on the other hand, that $V_{\underline{d}}(C_t) \to \psi_t(\mathfrak{J}_t^l)$ are locally trivial fibre bundles, with isomorphic fibres, for all $t \in \triangle \setminus \{o\}$. Then, the limit of the $V_{\underline{d}}(C_t)$'s is necessarily $\big(\mathscr{W}^o/T\big)|_{\mathcal{G}_o} = \mathcal{V}_{\underline{d}}(C_o)$.

(iii) Should be clear.
(iv) For the last statement, we notice that by taking the direct image over $(\mathbb{J}_o)^r$ of the exact sequence (B.1) we get the monomorphism of sheaves

$$0 \longrightarrow \mathcal{E}(C_o) = \oplus_\rho p_* \mathbb{L}_{d_\rho} \longrightarrow \mathrm{pr}^* \mathcal{W}(\tilde{C}_o) = \mathrm{pr}^* \big(\oplus_\rho p_* \tilde{\mathcal{L}}_{d_\rho} \big),$$

which is clearly T-equivariant. Moreover, since $\mathcal{E}(C_o)^o = \mathcal{E}(C_o) \cap \mathrm{pr}^* \mathcal{W}(\tilde{C}_o)^o$, the morphism

$$\mathbb{W}(C_o) := \mathcal{E}(C_o)^o / T \longrightarrow \mathcal{W}(\tilde{C}_o)^o / T := W(\tilde{C}_o)$$

is well-defined. On the other hand, $\nu^* \mathscr{W} = \mathcal{E}(C_o)$, and we find the morphism

$$\mathbb{W}(C_o) = \mathscr{E}(C_o)^o / T \longrightarrow \mathscr{W}^o / T = W(C_o).$$

Putting $\mathbb{V}_{\underline{d}}(C_o) := \mathbb{W}(C_o)|_{\mathbb{G}_o}$, we obtain the diagram

$$\begin{tikzcd}
 & & \mathbb{V}_{\underline{d}}(C_o) \arrow[dl, "\nu"'] \arrow[d] \arrow[dr, "N"] & \\
V_{\underline{d}}(C_o) \arrow[r, hook] \arrow[d] & \mathcal{V}_{\underline{d}}(C_o) \arrow[d] & \mathbb{G}_o \arrow[dl, "\nu"'] \arrow[dr, "\mathrm{pr}"] & V_{\underline{d}}(\tilde{C}_o) \arrow[d] \\
G_o \arrow[r, hook] & \mathcal{G}_o & & \tilde{G}_o,
\end{tikzcd} \tag{7.1}$$

and we notice that N sends a map $u : C_o \to X$ to $u \circ n : \tilde{C}_o \to X$, where $n : \tilde{C}_o \to C_o$ is the normalization. It is obviously biregular on its image over the Zariski dense open subset $\mathsf{Mor}_{\underline{d}}(C_o, X) \subset V_{\underline{d}}(C_o)$.

Finally, $\nu^* \Lambda_{\rho,o} \cong N^* \tilde{\Lambda}_\rho$ over $\mathbb{V}_{\underline{d}}(C_o)$ because both are associated to the same character. □

The definition of $\mathcal{V}_{\underline{d}}(C_o)$, as the limit of the varieties $V_{\underline{d}}(C_t)$ for $t \to o$, is very abstract: it uses the properness of a certain relative Hilbert scheme. This makes the definition unsatisfactory, if one wishes to have a picture of what happens during the deformation process. The following proposition is intended to fill in this gap, at least partially, and to clarify the geometric meaning of the points of $\mathcal{V}_{\underline{d}}(C_o)$. It is also used in section 9 to determine the components of $\mathcal{V}_{\underline{d}}(C_o)$ for Hirzebruch surfaces, which allows us to explicitely compute the corresponding Hamiltonian/gauge theoretical invariants.

We have pinned already down the component $V_{\underline{d}}(C_o)$, which is just compactification of $\mathsf{Mor}_{\underline{d}}(C_o, X)$; it projects onto G_o under the projection $\mathcal{V}_{\underline{d}}(C_o) \to \mathcal{G}_o$. We ask now for the geometric meaning of the union $V_{\underline{d}}^{\mathrm{bad}}(C_o)$ of the irreducible components of $\mathcal{V}_{\underline{d}}(C_o)$, other than $V_{\underline{d}}(C_o)$. First of all, it is pure dimensional, of dimension $|\underline{d}| - n(g-1)$, since $\mathcal{V}_{\underline{d}}(C_o)$ is so. Secondly, we have the projection $V_{\underline{d}}^{\mathrm{bad}}(C_o) \to G^{\mathrm{bad}}$, which is just a locally trivial toric fibration. And finally, we must clearly keep in mind that the points of $V_{\underline{d}}^{\mathrm{bad}}(C_o)$ are equivalence classes of r-tuples of sections in torsion free sheaves over C_o, which define, by the evaluative criterion of properness applied to X, morphisms from C_o (or $\tilde{C}_o$) to X whose multi-degree is not necessarily $\underline{d}$ anymore. As we have already mentioned, in this compactification we 'forget' the bubble components, but keep track of the bubbling points.

Proposition 7.2. *Let $K(C_o)$ be an irreducible component of $V_{\underline{d}}^{\mathrm{bad}}(C_o)$, and denote by $K \hookrightarrow G^{\mathrm{bad}}$ its projection.*

(i) *Then there is a primitive collection π having the property that for all points $j = (j_\rho)_\rho \in K$, the $\{j_\rho\}_{\rho\in\pi}$'s represent classes of non-locally free sheaves over C_o.*

(ii) *There is a multi-degree $\underline{d}_K = (d_{K,\rho})_\rho$ with the properties*
- *the $d_{K,\rho}$'s are positive,*
- $\mathrm{supp}(\underline{d}_K) := \{\rho \in \Sigma(1) \mid d_{K,\rho} > 0\} \supseteq \pi$,
- $\sum_{\rho\in\Sigma(1)} d_{K,\rho} e^\rho = 0$,

such that the multi-degree of the morphisms $\tilde{C}_o \to X$ corresponding to the general point of $K(C_o)$ is $\underline{d} - \underline{d}_K$. Moreover, these morphisms have the property that there is a 2-pointed stable map of multi-degree $\underline{d}_K$, defined on a tree of $\mathbb{P}^1$'s, which joins z' to z''.

Proof. (i) A consequence of the Gromov-compactness theorem is that in the limit $t \to o \in \triangle$ we should distinguish between two main possibilities:

Case 1. there is no bubbling at the double point of C_o.

Then the domain of definition of the stable maps which appear as the limit of morphisms $u_t : C_t \to X$, for $t \to o$, can be only of the form

$$C'_o = C_o \cup \left\{ \begin{array}{l} \text{trees of } \mathbb{P}^1\text{-s, attached at a finite} \\ \text{number of smooth points of } C_o \end{array} \right\}.$$

In terms of sections, it corresponds to equivalence classes of sections $[\underline{s}] = [(s_\rho)_\rho]$ having the property that $s_\rho \in \Gamma(C_o, L_\rho)$, with all the $L_\rho \to C_o$ locally free. The bubbling points correspond to smooth points $\zeta \in C_o$ for which there is some primitive collection π_0 such that $s_\rho(\zeta) = 0$, for all $\rho \in \pi_0$. Since ζ moves in $C_o \setminus \{z\}$, whose dimension is one, we deduce that the space of such sections has codimension at least $\#\pi_0 - 1 \geq 2 - 1 = 1$. In fact, these sections are limits of tuples of sections $(s_{\varepsilon,\rho})_\rho$ defining (honest) morphisms $C_o \to X$ of multi-degree $\underline{d}$. Consequently the corresponding points belong to $V_{\underline{d}}(C_o)$, the closure of $\mathsf{Mor}_{\underline{d}}(C_o, X)$ in $\mathcal{V}_{\underline{d}}(C_o)$.

Case 2. bubbling occurs at the double point of C_o.

In this case the domain of definition of the corresponding stable maps are of the form

$$C'_o = \tilde{C}_o \cup \left\{ \begin{array}{l} \text{a tree of } \mathbb{P}^1\text{-s joining} \\ \text{the points } z', z'' \in \tilde{C}_o \end{array} \right\} \cup \left\{ \begin{array}{l} \text{trees of } \mathbb{P}^1\text{-s, attached at a finite} \\ \text{number of points in } \tilde{C}_o \setminus \{z', z''\} \end{array} \right\}$$

and the restriction of such a stable map to $\tilde{C}_o$ is not defined on C_o anymore. We consider an irreducible component $K(C_o) \hookrightarrow V_{\underline{d}}^{\mathrm{bad}}(C_o)$, and let $K \hookrightarrow G^{\mathrm{bad}}$ be the

component onto which it projects. We consider a general point $j = (j_\rho)_\rho \in K$, and we assume that for each primitive collection π there is $\rho(\pi) \in \pi$ such that $j_{\rho(\pi)}$ represents the class of a locally free sheaf over C_o. We recall now that $K(C_o) \to K$ is a locally trivial toric bundle, and that the all our sheaves are globally generated. As the number of primitive collections is finite, for the general point $[\underline{s}] = [(s_\rho)_\rho] \in K(C_o)$ lying over j we have $\underline{s}(z'), \underline{s}(z'') \notin Z_X \hookrightarrow \mathbb{C}^r$ (when the s_ρ's are viewed as sections over $\tilde{C}_o$). But this means that bubbling appears neither at z' nor at z'', which is a contradiction. This proves our first claim for a general point $j \in K$, which is enough since the property is clearly preserved under specialisation.
(ii) Now we turn our attention to the second part of the proposition. It is clear that the multi-degree is locally constant on $K(C_o)$, and therefore constant on a Zariski dense open subset. Let $\underline{s} = (s_\rho)_\rho \in \oplus_\rho \Gamma(C_o, L_\rho)$ be a r-tuple of sections defining a general point of $K(C_o)$, and define

$$\tilde{L}_\rho := \begin{cases} n^*L_\rho/(n^*L_\rho)_{\mathrm{tor}} \text{ if } L_\rho \text{ is not locally free,} \\ n^*L_\rho \text{ if } L_\rho \text{ is locally free.} \end{cases}$$

The s_ρ's can be naturally viewed as sections in the line bundles $\tilde{L}_\rho \to \tilde{C}_o$ respectively, and $\deg \tilde{L}_\rho = d_\rho - 1$ for L_ρ not locally free; in particular this holds for $\rho \in \pi$. It is now obvious that the multi-degree of the morphism defined by $\underline{s}$ is componentwise smaller than $\underline{d}$, and that, of course, still obeys the linear relations (2.9). All together shows that the 'energy loss' $\underline{d}_K$ is componentwise positive, and fulfills the linear relation $\sum_\rho d_{K,\rho} e^\rho = 0$.

The statement about the existence of the tree of $\mathbb{P}^1$'s joining z' and z'' is just a consequence of the Gromov-compactness theorem. □

Lemma 7.2. *Let $K \hookrightarrow \mathcal{G}_o$ be an irreducible component, and consider the irreducible components $\mathbb{K}', \mathbb{K}'' \hookrightarrow \nu^{-1}K \hookrightarrow \mathbb{G}_o$. We define $\mathbb{K}'(C_o) := \mathbb{V}_{\underline{d}}(C_o)|_{\mathbb{K}'}$ and similarly for $\mathbb{K}''$.*

(i) *Then $N_*\mathbb{K}'(C_o), N_*\mathbb{K}''(C_o) \hookrightarrow V_{\underline{d}}(\tilde{C}_o)$ define the same class in the Chow group, modulo homological equivalence.*
(ii) *If $\mathbb{K} \hookrightarrow \nu^{-1}K$ is an irreducible component, then $N_*[\mathbb{K}(C_o)] \in A^*(V_{\underline{d}}(\tilde{C}_o))$ uniquely corresponds to a class $c_{\mathbb{K}} \in A^*(G_{\tilde{C}_o}) \otimes A^*_T$ of degree* $\dim X$.

Proof. (i) For a general point $j = (j_\rho)_\rho \in K \hookrightarrow \mathcal{J}^r_o$, we split $\Sigma(1)$ into the disjoint union $\Sigma_{\mathrm{free}} \cup \Sigma_{\mathrm{bad}}$, according to whether j_ρ represent the class of a locally free sheaf over C_o or not. The normalization map ν is biregular over the locus of locally free sheaves over C_o, and it is finite over its complement. The possibility of appearing several irreducible components in $\nu^{-1}K$ comes precisely from the finiteness of ν over the coordinates Σ_{bad}. If $\mathscr{J}'$ and $\mathscr{J}''$ are general points of $\mathbb{K}'$ and $\mathbb{K}''$ respectively, both lying over j, we notice that the Σ_{free}-coordinates of $N(\mathscr{J}'), N(\mathscr{J}'') \in \tilde{J}_o^{\Sigma(1)}$ coincide, while the other coordinates differ by a translation through an element of $\tau \in \big(\mathrm{Pic}^0(\tilde{C}_o)\big)^{\Sigma_{\mathrm{bad}}}$, which is just a tuple of some powers of $\mathcal{O}_{\tilde{C}_o}(z' - z'')$; τ is clearly independent on the choice of the (general) point $j \in K$. On the other hand, the fibres $\mathbb{K}'(C_o)_{\mathscr{J}'}$

and $\mathbb{K}''(C_o)_{\mathscr{Y}''}$ are naturally identified with $K(C_o)_j$. All together we find that we are in the following situation

$$\begin{tikzcd}
N_*\mathbb{K}''(C_o) = \tau^* N_*\mathbb{K}'(C_o) \arrow[r] \arrow[d, hook] & N_*\mathbb{K}'(C_o) \arrow[d, hook] \\
\tilde{J}_o^{\Sigma(1)} \arrow[r, "\tau"] & \tilde{J}_o^{\Sigma(1)}
\end{tikzcd}$$

and therefore $N_*\mathbb{K}'(C_o)$ and $N_*\mathbb{K}''(C_o)$ define the same class in $V_{\underline{d}}(\tilde{C}_o)$ modulo homological equivalence.

(ii) We have proved in lemma 7.1 that for each irreducible component $\mathbb{K} \hookrightarrow \mathbb{G}_o$, the projection $\mathbb{K} \to \tilde{G}_o$ is surjective. Then $N_*\mathbb{K}(C_o) \hookrightarrow V_{\underline{d}}(\tilde{C}_o)$ is irreducible, projects onto $\tilde{G}_o$, and defines an element in $A^n\big(V_{\underline{d}}(\tilde{C}_o)\big)$. We recall now that the components of $\underline{d}$ were chosen large enough, and therefore

$$A^n\big(V_{\underline{d}}(\tilde{C}_o)\big) \cong A^n_T\big(\mathscr{V}_{\underline{d}}(\tilde{C}_o)^o\big) \overset{4.1}{\cong} A^n_T\big(\mathscr{V}_{\underline{d}}(\tilde{C}_o)\big) \cong A^n_T(G_{\tilde{C}_o}) \\ \cong \big[A^*(G_{\tilde{C}_o}) \otimes A^*_T\big]_{(n)}.$$

The surjectivity of $N_*\mathbb{K}(C_o) \to G_{\tilde{C}_o}$ translates into the fact that the A^n_T-component of this class does not vanish. □

In most cases it is difficult to describe the components of $V^{\mathrm{bad}}_{\underline{d}}(C_o)$ in more detail, the difficulty arising from the presence of the tree of $\mathbb{P}^1$'s joining z' and z'', which is hard to control. However, there is one case when things become simpler, namely when the tree reduces to a morphism $\mathbb{P}^1 \to X$. In view of well-known results in the theory of J-holomorphic curves in symplectic geometry, this should be always possible for semi-positive, or at least for Fano, toric varieties X. But this is an issue that we are unable to address here.

We have already seen in lemma 3.1 that, for a primitive family π, the vanishing of the π-components of a tuple of sections implies that the multi-degree of the induced morphism drops by $\underline{d}_\pi$. Then, in view of point (ii) of the previous proposition, it is natural to try describing that component of $V^{\mathrm{bad}}_{\underline{d}}(C_o)$ whose general points induce morphisms $v : \tilde{C}_o \to X$ of multi-degree $\underline{d} - \underline{d}_\pi$, with the property that there is a morphism $\mathbb{P}^1 \to X$ of degree $\underline{d}_\pi$ joining $v(z')$ and $v(z'')$. We are going to denote by $V_{\underline{d},\pi}(C_o)$ this component. Then there is a well-defined morphism $\nu^{-1}V_{\underline{d},\pi}(C_o) \to V_{\underline{d}}(\tilde{C}_o)$ which factorises through

$$\begin{tikzcd}
\nu^{-1}V_{\underline{d},\pi}(C_o) \arrow[rr, hook] \arrow[dr, hook] & & V_{\underline{d}}(\tilde{C}_o) \\
& V^{z'\leftrightarrow z''}_{\underline{d},\pi}(\tilde{C}_o, z') \cup V^{z'\leftrightarrow z''}_{\underline{d},\pi}(\tilde{C}_o, z'') \arrow[ur, hook] &
\end{tikzcd}$$

Here $V^{z'\leftrightarrow z''}_{\underline{d},\pi}(\tilde{C}_o, z')$ denotes the locus of those points in $V_{\underline{d},\pi}(\tilde{C}_o)$ (see (3.8) for the definition) whose π-components vanish at z', and moreover z' and z'' can be joined by a morphism $\mathbb{P}^1 \to X$ of multi-degree $\underline{d}_\pi$. We use similar notation for z'' in place of z'. Our goal now is to describe these varieties.

For answering this issue, and for later purposes too, we wish to recall from (5.2) that we have the evaluation morphism

$$\begin{tikzcd} \Phi : \big(\mathcal{P}_{\underline{\delta}} \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T \arrow[r] \arrow[d] & \mathbb{C}^r \\ \tilde{C}_o \times V_{\underline{d}}(\tilde{C}_o) & \end{tikzcd}$$

The restrictions $\mathcal{P}' := \mathcal{P}_{\underline{\delta}}|_{\{z'\}\times\mathcal{J}^l}$ and $\mathcal{P}'' := \mathcal{P}_{\underline{\delta}}|_{\{z''\}\times\mathcal{J}^l}$ determine the morphism $\Psi = (\Psi_{z'}, \Psi_{z''})$

$$\begin{tikzcd} \big((\mathcal{P}' \times_{\mathcal{J}^l} \mathcal{P}'') \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T \arrow[r, "\Psi"] \arrow[d, "T\times T"'] & \mathbb{C}^r \times \mathbb{C}^r \arrow[d, dashed] \\ V_{\underline{d}}(\tilde{C}_o) \arrow[r, dashed, "ev_{z'z''}"] & X \times X \end{tikzcd} \tag{7.2}$$

which covers the evaluation at z' and z''.

The assumption $d_\rho \geq 2g + 1$ for all ρ implies that Ψ is submersive, and therefore it is flat; its fibres are smooth and irreducible, since $\big((\mathcal{P}' \times_{\mathcal{J}^l} \mathcal{P}'') \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T$ is so. There is an induced *ring homomorphism*

$$\begin{aligned} \Psi^*_{T\times T} : A^T_* \otimes A^T_* \cong A^{T\times T}_*(\mathbb{C}^r \times \mathbb{C}^r) \longrightarrow A^{T\times T}_*\Big[\big((\mathcal{P}' \times_{\mathcal{J}^l} \mathcal{P}'') \times_{\mathcal{J}^l} \mathcal{V}^o\big)/T\Big] \\ \cong A_* V_{\underline{d}}(\tilde{C}_o), \end{aligned}$$

where for the last isomorphism we use [9, theorem 4]. We have denoted by A^T_* the T-equivariant Chow ring of a point; according to [9, section 3.2], the natural cycle map $A^T_* = A^T_*(point) \to H^T_*(point) =: H^T_*$ is an isomorphism.

The analogous evaluation morphism for maps $\mathbb{P}^1 \to X$ having multi-degree $\underline{d}_\pi$, takes the following very down-to-earth form

$$\Big[\bigoplus_{\rho\notin\mathcal{R}} \mathbb{C}\Big] \oplus \Big[\bigoplus_{\rho\in\mathcal{R}} \mathcal{O}_{\mathbb{P}^1}(-d_{\pi,\rho}) \otimes \Gamma\big(\mathbb{P}^1, \mathcal{O}(d_{\pi,\rho})\big)\Big] \longrightarrow \mathbb{C}^{\Sigma(1)}, \text{ for } \mathcal{R} := \operatorname{supp}(\underline{d}_\pi).$$

By fixing $(0, 1), (1, 0) \in \mathbb{C}^2$ above $[0, 1], [1, 0] \in \mathbb{P}^1$ respectively, the evaluation morphism at these points is

$$\bigoplus_{\rho\notin\mathcal{R}} \mathbb{C} \oplus \bigoplus_{\rho\in\mathcal{R}} \Gamma\big(\mathbb{P}^1, \mathcal{O}(d_{\pi,\rho})\big) \longrightarrow \mathbb{C}^{\Sigma(1)} \oplus \mathbb{C}^{\Sigma(1)} \xrightarrow{\mathrm{pr}_\pi} \mathbb{C}^{\Sigma(1)\setminus\mathcal{R}} \oplus \mathbb{C}^{\Sigma(1)\setminus\mathcal{R}}, \tag{7.3}$$

$$\big((c_\rho)_{\rho\notin\mathcal{R}}, (\ell_\rho)_{\rho\in\mathcal{R}}\big) \longmapsto \big((c_\rho)_{\rho\notin\mathcal{R}}, (\ell_\rho(1, 0))_{\rho\in\mathcal{R}}\big) \oplus \big((c_\rho)_{\rho\notin\mathcal{R}}, (\ell_\rho(0, 1))_{\rho\in\mathcal{R}}\big).$$

Comparing now the diagrams (7.2) with (7.3), we can answer which are the morphisms $\tilde{C}_o \to X$ for which z' and z'' can be joined by a $\mathbb{P}^1$.

Lemma 7.3. *Assume that π is a primitive family, and that the multi-degree $\underline{d}$ satisfies $d_\rho \geq 2g + 1$ for all $\rho \in \Sigma(1)$, then*

$$[V^{z'\leftrightarrow z''}_{\underline{d},\pi}(\tilde{C}_o, z')] = \Psi^*\Big[\mathrm{pr}^*_\pi\big(\overline{(T\times T)\cdot\Delta_{\mathbb{C}^{\Sigma(1)\setminus\mathcal{R}}}}\big)\cdot\big(\textstyle\prod_{\rho\in\pi}\chi_\rho\otimes 1\big)\Big] \in A_*\big(V_{\underline{d}}(\tilde{C}_o)\big),$$

where $\Delta_{\mathbb{C}^{\Sigma(1)\setminus\mathcal{R}}} \hookrightarrow \mathbb{C}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{C}^{\Sigma(1)\setminus\mathcal{R}}$ *is the diagonal. The co-dimension of this class equals* $r - \big(|\mathcal{R}| - |\pi| + \dim\langle\chi_\rho \mid \rho\in\Sigma(1)\setminus\mathcal{R}\rangle\big)$.

Proof. Is clear from (7.3) above that, for joining z' to z'', there are no restrictions on the $\mathcal{R}$-components of the evaluation Ψ; the only restriction is that the remaining components are in the $T\times T$-closure of the diagonal. What it remains to do is to further intersect with the class defining $V_{\underline{d},\pi}(\tilde{C}_o) \hookrightarrow V_{\underline{d}}(\tilde{C}_o)$, namely with $\prod_{\rho\in\pi}\Lambda_\rho$. □

We wish to use propositions 7.1 and 7.2 for finding recursive formulae for the Hamiltonian invariants. In the case of the usual Gromov-Witten invariants, one proceeds as follows: the restrictions at z' and z'' of the evaluation (5.1) determines the *rational map*

$$ev_{z'z''} : V_{\underline{d}}(\tilde{C}_o) \dashrightarrow X \times X,$$

and the class that we are looking for is the pull-back $ev^*_{z'z''}[\Delta_X]$, where $\Delta_X \hookrightarrow X\times X$ is the diagonal. The shortcoming of this reasoning relies in the fact that $ev_{z'z''}$ is not everywhere defined. More precisely,

$$[\Delta_X] = \sum_\kappa h'_\kappa \otimes h''_\kappa \in H^*(X;\mathbb{Z})\otimes H^*(X;\mathbb{Z}),$$

where $(h'_\kappa)_\kappa$ and $(h''_\kappa)_\kappa$ are dual bases, but it is almost impossible to explicitly express the pull-backs of $h'_\kappa\otimes h''_\kappa$ in terms of the $\tilde{\Lambda}_\rho \to V(\tilde{C}_o)$'s since $ev^*_{z'z''}$ is not a ring homomorphism. For this reason we are forced to adopt a different strategy, and use the *evaluation morphism* (5.2).

Theorem 7.1. *Let X be a smooth toric variety, and consider an arbitrary multi-degree $\underline{d} = (d_\rho)_\rho$ which obeys the linear relations* (2.9). *For any equivariant class $a \in A^T_* \cong A^*_T$ of degree $|\underline{d}| - \dim X\cdot(g-1)$, the following formula holds*

$$I^g_{\underline{d}}(a) = I^{g-1}_{\underline{d}}(c_g\cdot a),$$

where $c_g \in A^*(G_{\tilde{C}})\otimes A^*_T$ is an equivariant class of degree $\dim X$, *which does not depend on $\underline{d}$. Here $\tilde{C}$ denotes a smooth curve of genus $g-1$.*

Proof. Let us prove first that, in the case that such a formula holds, the class c_g does not depend on $\underline{d}$. So, let us consider arbitrary multi-degrees $\underline{d}$ and $\underline{e}$, with $\underline{e}\geq 0$ (componentwise). Using the recursive relation (5.3) we find that

$$I^{g-1}_{\underline{d}}(c_{g,\underline{d}}\cdot a) = I^g_{\underline{d}}(a) = I^g_{\underline{d}+\underline{e}}\big(\chi^{\underline{e}}\cdot a\big) = I^{g-1}_{\underline{d}+\underline{e}}\big(c_{g,\underline{d}+\underline{e}}\cdot\chi^{\underline{e}}\cdot a\big) = I^{g-1}_{\underline{d}}(c_{g,\underline{d}+\underline{e}}\cdot a),$$

and therefore $c_{g,\underline{d}} = c_{g,\underline{d}+\underline{e}}$. Since $\underline{e}\geq 0$ was taken arbitrary, we deduce that c_g is independent of $\underline{d}$. Strictly speaking, this argument shows only that *we may choose* a class $c_g \in A^*(G_{\tilde{C}})\otimes A^*_T$ which does not depend on $\underline{d}$. In order to prove that the classes which appear for different degrees are actually equal, we should degenerate the embedding $V_{\underline{d}}(C) \hookrightarrow V_{\underline{d}+\underline{e}}(C)$ defined in (3.6), as $C \rightsquigarrow C_o$; however, a moment's thought shows that things turn out the right way.

We prove now the existence of the class c_g; it suffices to prove the equality for $a = \chi_1^{m_1} \cdot \ldots \cdot \chi_r^{m_r}$, with $(m_\rho)_\rho$ positive integers such that $\sum_\rho m_\rho = |\underline{d}| - n(g-1)$, since all elements in A_T^* are linear combinations of such products.

Let us choose a smooth curve C of genus g and a degeneration $\mathscr{C} \to \triangle$ to an irreducible curve C_o having precisely one node, and we denote by $\mathcal{K}$ the set of the irreducible components of $\mathcal{G}_o$. According to proposition 7.1, the elements of $\mathcal{K}$ correspond bijectively to the irreducible components of $\mathcal{V}_{\underline{d}}(C_o)$, and $\mathcal{K}$ contains the distinguished element defined by G_o which corresponds to $V_{\underline{d}}(C_o)$.

For $K \in \mathcal{K}$, let ν_K be the number of irreducible components of $\nu^{-1}K$; $K(C_o)$ will stay for the subscheme of $\mathcal{V}_{\underline{d}}(C_o)$ determined by K, that is $K(C_o) := K \times_{\mathcal{G}_o} \mathcal{V}_{\underline{d}}(C_o)$.

We start by considering the case when $d_\rho \geq \max\left\{\frac{n}{2} + g - 1, 2g+1\right\}$ for all ρ: then the virtual fundamental class of $V_{\underline{d}}(C)$ coincides with its fundamental class.

$$\begin{aligned} I_{\underline{d}}^g(m_1, \ldots, m_r) &= \int_{V_{\underline{d}}(C_t)} \Lambda_{1,t}^{m_1} \cdot \ldots \cdot \Lambda_{r,t}^{m_r} = \int_{\mathcal{V}_{\underline{d}}(C_o)} \Lambda_{1,o}^{m_1} \cdot \ldots \cdot \Lambda_{r,o}^{m_r} \\ &= \sum_{K \in \mathcal{K}} \int_{K(C_o)} \Lambda_{1,o}^{m_1} \cdot \ldots \cdot \Lambda_{r,o}^{m_r} = \sum_{K \in \mathcal{K}} \frac{1}{\nu_K} \int_{\nu^{-1}K(C_o)} N^*\big(\tilde{\Lambda}_1^{m_1} \cdot \ldots \cdot \tilde{\Lambda}_r^{m_r}\big) \\ &= \sum_{K \in \mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K} \hookrightarrow \nu^{-1}K} \int_{\mathbb{K}(C_o)} N^*\big(\tilde{\Lambda}_1^{m_1} \cdot \ldots \cdot \tilde{\Lambda}_r^{m_r}\big) \\ &= \sum_{K \in \mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K} \hookrightarrow \nu^{-1}K} \int_{N_*\mathbb{K}(C_o)} \tilde{\Lambda}_1^{m_1} \cdot \ldots \cdot \tilde{\Lambda}_r^{m_r} \\ &\overset{7.2}{=} \sum_{K \in \mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K} \hookrightarrow \nu^{-1}K} \int_{V_{\underline{d}}(\tilde{C}_o)} c_{\mathbb{K}} \cdot \tilde{\Lambda}_1^{m_1} \cdot \ldots \cdot \tilde{\Lambda}_r^{m_r}. \end{aligned} \tag{7.4}$$

We conclude that

$$c_g = \sum_{K \in \mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K} \hookrightarrow \nu^{-1}K} c_{\mathbb{K}} \tag{7.5}$$

has the desired property, and we deduce from lemma 7.2 (i) that this class is actually integral.

We prove now our statement for arbitrary multi-degrees: let $\underline{d}$ be such, and we choose a multi-degree $\underline{d}' = (d'_\rho)_\rho$ which still obeys the linear relations (2.9), $d'_\rho \geq 2g+1$, and $e_\rho := d'_\rho - d_\rho > 0$ for all ρ. We have already seen in the proof of the proposition 5.1 that this data induces an embedding $V_{\underline{d}}(C) \hookrightarrow V_{\underline{d}'}(C)$ and that $[\![V_{\underline{d}}(C)]\!] = \big(\prod_\rho \Lambda_\rho^{e_\rho}\big) \cap [V_{\underline{d}'}(C)]$. Then for $a \in A_T^*$ we obtain

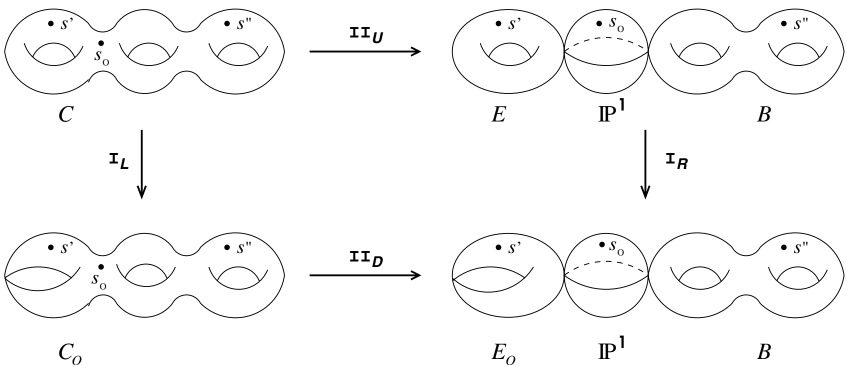
$$I_{\underline{d}}^g(a) \overset{(5.3)}{=} I_{\underline{d}'}^g(\chi_1^{e_1} \cdot \ldots \cdot \chi_r^{e_r} \cdot a) = I_{\underline{d}'}^{g-1}(c_g \cdot \chi_1^{e_1} \cdot \ldots \cdot \chi_r^{e_r} \cdot a) \overset{(5.3)}{=} I_{\underline{d}}^{g-1}(c_g \cdot a).$$

This concludes the proof of the theorem. □

8. Degenerations: Part II

In the previous section we have seen that the Hamiltonian invariants fulfill a simple recursion formula, which is very satisfactory except that the class appearing there is rather mysterious. However, there is a hint indicating that the class c_g should be independent of the genus: the degeneration of the l^{th} power of the Jacobian variety, which is crucial for defining c_g, looks – in a very naive sense – the same in all genera.

It turns out that the right approach to this idea is to relate the degeneration for curves of genus g, studied in the previous section, to the degeneration of an elliptic curve to a rational, nodal curve; this leads to consider a 'commutative diagram' of the type We notice that in this diagram appear the first type degenerations in genera



g and 1 respectively, used for obtaining the recursive formula 7.1, and the second type degeneration 'interpolates' between the degenerations $C \rightsquigarrow C_o$ and $E \rightsquigarrow E_o$.

We start working in a slightly more general context: we fix two smooth and irreducible curves C' and C'' of genus g' and g'' respectively, and we choose furthermore two multi-degrees $\underline{d}'$ and $\underline{d}''$ satisfying (2.9). We define $g = g' + g''$ and $\underline{d} := \underline{d}' + \underline{d}''$; unless otherwise stated, we will further assume that $d'_\rho \geq \max\left\{\frac{n}{2} + g', 2g' + 1\right\}$ and $d''_\rho \geq \max\left\{\frac{n}{2} + g'', 2g'' + 1\right\}$ for all ρ.

We consider the curve $C' + C''$ obtained by joining C' and C'' with a $\mathbb{P}^1$, and our first goal is to compactify the space of morphisms $\mathsf{Mor}_{\underline{d}',\underline{d}''}(C' + C'', X)$ whose restrictions to C', $\mathbb{P}^1$ and C'' have multi-degrees $\underline{d}'$, $\underline{0}$ and $\underline{d}''$ respectively. We denote by $\mathfrak{J}_{C'}$ the Jacobian variety of C', and by $\mathfrak{J}_{C''}$ that of C''. Theorem B.3 says that the Poincaré bundle over $(\mathfrak{J}_{C'} \times \mathfrak{J}_{C''}) \times (C' + C'')$ exists: it parameterizes line bundles on $C' + C''$ whose restriction to C', C'' and $\mathbb{P}^1$ have degree d'_ρ, d''_ρ and 0 respectively. We denote $\mathcal{W}_\rho$ the corresponding Picard bundle over $\mathfrak{J}_{C'} \times \mathfrak{J}_{C''}$, and let $\mathcal{W}_{\underline{d}',\underline{d}''}(C' + C'') := \oplus_\rho \mathcal{W}_\rho \to \mathfrak{J}^r_{C'} \times \mathfrak{J}^r_{C''}$.

Similarly, there is a morphism $(\psi_{C'}, \psi_{C''}) : \mathfrak{J}^l_{C'} \times \mathfrak{J}^l_{C''} \to \mathfrak{J}^r_{C'} \times \mathfrak{J}^r_{C''}$ defined as in (2.12), and its image is $G_{C'} \times G_{C''}$. We denote now $\mathcal{V}_{\underline{d}',\underline{d}''}(E + B) := \mathcal{W}_{\underline{d}',\underline{d}''}(E + B)|_{G_{C'} \times G_{C''}}$.

Now we repeat word by word the construction performed in section 3, in the case of a smooth and irreducible curve: for each primitive collection π of the fan Σ defining X, we let $\mathcal{V}(\pi) \hookrightarrow \mathcal{V}_{\underline{d}',\underline{d}''}(C' + C'')$ be the sub-vector bundle whose π-components vanish, and define $\mathcal{V}_{\underline{d}',\underline{d}''}(C'+C'')^o := \mathcal{V}_{\underline{d}',\underline{d}''}(C'+C'') \setminus \bigcup_\pi \mathcal{V}(\pi)$.

Proposition 8.1. (i) *The quotient* $V_{\underline{d}',\underline{d}''}(C'+C'') := \mathcal{V}_{\underline{d}',\underline{d}''}(C'+C'')^o \big/ T$ *exists, and it is proper over* $\mathfrak{J}^r_{C'} \times \mathfrak{J}^r_{C''}$*; its fibres are smooth and projective toric varieties. It is a smooth and projective compactification of* $\mathsf{Mor}_{\underline{d}',\underline{d}''}(C' + C'', X)$.

(ii) *The natural morphism* $b : \mathcal{V}_{\underline{d}',\underline{d}''}(C' + C'') \to \mathcal{V}_{\underline{d}''}(C')$ *is flat, surjective and* T*-equivariant, and it induces the rational map* $\hat{b} : V_{\underline{d}',\underline{d}''}(C' + C'') \dashrightarrow V_{\underline{d}''}(C')$.

Proof. Both statements are obvious. □

Step 1. the degeneration $\mathtt{II}_U$. We choose now a curve $\triangle \to \overline{M}_{g,3}$ in the Deligne-Mumford space, with the property that there is $o \in \triangle$ which is sent to $[C' + C''] \in \overline{M}_{g,3}$ and $\triangle \setminus \{o\} \subset M_{g,3}$. This path corresponds to a flat family $\mathscr{C} \to \triangle$ of curves of genus g, over a 1-dimensional base, whose general fibre is smooth and irreducible, and the special fibre over $o \in \triangle$ is the curve $C' + C''$ described above. Moreover, this family has three pairwise disjoint sections $s', s'', s_0 : \triangle \to \mathscr{C}$ passing through the smooth locus of the fibres of $\mathscr{C} \to \triangle$.

Theorem B.4 says that in this case the relative compactified Jacobian $\mathscr{J} \to \triangle$ exists, and its central fibre is $\mathfrak{J}_o = \mathfrak{J}_{C'} \times \mathfrak{J}_{C''}$. For each ρ, there is a Poincaré bundle $\mathscr{L}_\rho \to \mathscr{J} \times_\triangle \mathscr{C}$ which is trivialised along $\mathscr{J} \times_\triangle s_0(\triangle)$, and which parameterizes line bundles of 3-degree $(d'_\rho, 0, d''_\rho)$ over $C' + C''$. Repeating *ad litteram* the construction done in 5.1 and 7.1, we deduce that there is a flat family $\mathcal{V}_{\underline{d}}(\mathscr{C}) \to G_{\mathscr{C}} \to \triangle$ whose general fibre over $t \in \triangle$ is $V_{\underline{d}}(C_t)$, and the special fibre is $V_{\underline{d}',\underline{d}''}(C' + C'') \to G_{C'} \times G_{C''}$ constructed in 8.1 above. This implies that for $a \in A^*_T$,

$$I^g_{\underline{d}}(a) = \int_{V_{\underline{d}',\underline{d}''}(C'+C'')} \Phi^*_T(a)/[s_0] =: I^{g',g''}_{\underline{d}',\underline{d}''}(a).$$

Step 2. the degeneration $\mathtt{I}_R$. Now we are going to restrict ourselves to the situation illustrated in the picture. We have to choose now a family of elliptic curves which degenerate to a rational, nodal curve: this corresponds to a path $\triangle \to \overline{M}_{1,2}$, with the property that there is $o \in \triangle$ which is mapped to $[E_o] \in \overline{M}_{1,2}$, and $\triangle \setminus \{o\} \subset M_{1,2}$. This way we get a flat family $\mathscr{E} \to \triangle$ of curves of genus 1, over a 1-dimensional base, together with two disjoint sections $Q', s' : \triangle \to \mathscr{E}$ passing through the smooth locus of $\mathscr{E} \to \triangle$. Theorem B.2 applies, and we deduce the existence of the relative compactified Jacobian $\mathscr{J} \to \triangle$, and of the relative Poincaré sheaf $\mathscr{L}' \to \mathscr{J} \times_\triangle \mathscr{E} \to \triangle$, trivialised along the section Q'.

We fix a curve B of genus $g - 1$, and two points Q'', s'' on it, and consider the Poincaré bundle $\mathcal{L}'' \to \mathfrak{J}_B \times B$, trivialised at Q''.

Joining Q' with Q'' using a $\mathbb{P}^1$, we obtain a flat degeneration $\mathscr{E} + B \to \triangle$, and the same construction as in theorem B.3 provides us with a Poincaré sheaf

$\mathscr{L} \to (\mathscr{J} \times \mathfrak{J}_B) \times_{\triangle} (\mathscr{E} + B) \to \triangle$ which is trivialised along the section defined by the point $s_0 \in \mathbb{P}^1$. For each ρ we denote $\mathscr{L}_\rho$ the relative Poincaré sheaf of 3-degree $(d'_\rho, 0, d''_\rho)$, and by $\mathscr{W}_\rho \to \mathscr{J} \times \mathfrak{J}_B$ the corresponding Picard bundle. For $\mathscr{W} := \oplus_\rho \mathscr{W}_\rho$, we find again the 'good open subset' $\mathscr{W}^o = \mathscr{W} \setminus \bigcup_\pi \mathscr{W}(\pi)$ for which the quotient $\mathscr{W}^o/T$ exists, and is proper over $\triangle$.

The morphisms $(\psi_t, \psi_B) : \mathfrak{J}^l_{E_t} \times \mathfrak{J}^l_B \to \mathfrak{J}^r_{E_t} \times \mathfrak{J}^r_B$, for $t \neq o$, patch together and define a rational map $(\psi, \psi_B) : \mathscr{J}^l \times \mathfrak{J}^l_B \dashrightarrow \mathscr{J}^r \times \mathfrak{J}^r_B$. Using the lemma 7.1, but in genus 1 now, we find that the limit as $t \to o$ of the images of the morphisms (ψ_t, ψ_B) exists and equals $\mathcal{G}_o \times G_B \hookrightarrow \mathfrak{J}^r_o \times \mathfrak{J}^r_B$. We define $\mathscr{V} := \mathscr{W}|_{\overline{\mathrm{Image}(\psi, \psi_B)}}$ and $\mathscr{V}^o := \mathscr{V} \cap \mathscr{W}^o$. In complete analogy with proposition 7.1 we have

Proposition 8.2. (i) *The quotient $\mathscr{V}^o/T$ exists and is projective over $\triangle$; its fibre over $t \in \triangle \setminus \{o\}$ coincides with $V_{\underline{d}',\underline{d}''}(E_t + B)$ constructed in 8.1.*

(ii) *The fibre $\mathcal{V}_{\underline{d}',\underline{d}''}(E_o + B)$ of $\mathscr{V}^o/T$ over $o \in \triangle$ is the limit of the subvarieties $V_{\underline{d}',\underline{d}''}(E_t + B) \hookrightarrow \mathscr{W}^o_t/T$ in a suitable relative Hilbert scheme. Denoting $\mathfrak{V}_{\underline{d}',\underline{d}''}(E_o+B) := \mathscr{V}|_{\mathcal{G}_o \times G_B}$, we have $\mathcal{V}_{\underline{d}',\underline{d}''}(E_o+B) = \mathfrak{V}_{\underline{d}',\underline{d}''}(E_o+B)^o/T$.*

(iii) *There is the following diagram*

$$\begin{tikzcd}[column sep=small]
 & \mathbb{V}_{\underline{d}',\underline{d}''}(E_o + B) \arrow[dl, "{(\nu,\mathrm{id})}"'] \arrow[d] \arrow[dr, "{(N,\mathrm{id})}"] & \\
\mathcal{V}_{\underline{d}',\underline{d}''}(E_o + B) \arrow[d] & \mathbb{G}_o \times G_B \arrow[dl, "{(\nu,\mathrm{id})}"'] \arrow[dr, "{(\mathrm{pr},\mathrm{id})}"] & V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B) \arrow[d] \\
\mathcal{G}_o \times G_B & & G_{\tilde{E}_o} \times G_B \cong G_B,
\end{tikzcd} \tag{8.1}$$

where $\tilde{E}_o \cong \mathbb{P}^1$ denotes the normalization of E_o.

An immediate consequence is that

$$I^{1,(g-1)}_{\underline{d}',\underline{d}''}(a) = \int_{\mathcal{V}_{\underline{d}',\underline{d}''}(E_o+B)} \Phi^*_T(a)/[s_0].$$

The irreducible components of $\mathcal{G}_o \times G_B$ are in one-to-one correspondence with those of $\mathcal{G}_o$, and their multiplicities coincide too. To each irreducible component $K \hookrightarrow \mathcal{G}_o$ corresponds a component $K_{\underline{d}',\underline{d}''}(E_o + B) \hookrightarrow \mathcal{V}_{\underline{d}',\underline{d}''}(E_o + B)$, and vice versa. In order to relate the invariants corresponding to $E_o + B$ to those corresponding to $\tilde{E}_o + B$, we must make the same game as in the proof of theorem 7.1; more precisely, for an irreducible component $\mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B) \hookrightarrow (\nu, \mathrm{id})^{-1} K_{\underline{d}',\underline{d}''}(E_o + B)$, we must compare

$$\big[(N, \mathrm{id})_* \mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B)\big] \in A_*\big(V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)\big) \text{ with } \big[N_* \mathbb{K}_{\underline{d}'}(E_o)\big] \in A_*\big(V_{\underline{d}'}(\tilde{E}_o)\big).$$

The result in this direction is:

Proposition 8.3. (i) $\big[(N, \mathrm{id})_* \mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B)\big] \in A_*\big(V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)\big)$ *is the image of* $\big[N_* \mathbb{K}_{\underline{d}'}(E_o)\big] \in A_*\big(V_{\underline{d}'}(\tilde{E}_o)\big)$ *under the correspondence defined by* $\hat{b}$.

(ii) *In theorem 7.1, we may take c_g to be equal to a class $c \in A^n_T$ which is independent of the genus g.*

Proof. (i) We start noticing that $\hat{b}$ is defined on the Zariski open subset

$$V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)^o := \left\{ [\underline{s}] \,\middle|\, \begin{array}{l} \underline{s} \in H^0(\tilde{E}_o + B, \oplus_\rho L_\rho) \\ \mathrm{Image}(\underline{s}|_{\tilde{E}_o}) \not\subset \bigcup_\pi \mathbb{A}(\pi) \end{array} \right\} \subset V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)$$

whose complement has codimension strictly larger than n; consequently, the Chow groups of degree n of $V_{\underline{d}',\underline{d}''}(\tilde{E}_o+B)^o$ and $V_{\underline{d}',\underline{d}''}(\tilde{E}_o+B)$ are isomorphic. In the diagram

$$\begin{array}{ccc}
\mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B) & \xrightarrow{(N,\mathrm{id})} & V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B) \\
\cup & & \cup \\
\mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B)^o & \xrightarrow{(N,\mathrm{id})} & V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)^o \\
\downarrow & & \downarrow \hat{b} \ \text{smooth fibration} \\
\mathbb{K}_{\underline{d}'}(E_o) & \xrightarrow{N} & V_{\underline{d}'}(\tilde{E}_o)
\end{array}$$

the lower square is cartesian, and, using [12, proposition 1.7], we deduce that $(N, \mathrm{id})_*\mathbb{K}_{\underline{d}',\underline{d}''}(E_o + B)^o = \hat{b}^{-1}\big(N_*\mathbb{K}_{\underline{d}'}(E_o)\big)$, as cycles in $V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)^o$. The conclusion follows now from the previous remark.

(ii) We notice that if we degenerate an elliptic curve to a rational nodal curve, the the cycle $N_*\mathbb{K}_{\underline{d}'}(E_o)$ defines an element in $A^n\big(V_{\underline{d}'}(\tilde{E}_o)\big)$, which, according to remark 4.1, is naturally isomorphic to A^n_T. Let us denote $c_{1,\mathbb{K}}$ this class. Equality (7.5) says that

$$c_1 = \sum_{K\in\mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K}\hookrightarrow\nu^{-1}K} c_{1,\mathbb{K}},$$

where $\mathcal{K}$ denotes the set of the irreducible components of $\mathcal{G}_o$. On the other hand, the correspondence

$$\hat{b}^* : A^n_T \cong A^n\big(V_{\underline{d}'}(\tilde{E}_o)\big) \to A^n\big(V_{\underline{d}',\underline{d}''}(\tilde{E}_o + B)\big) \cong \big[A^*(G_B) \otimes A^*_T\big]_{(n)}$$

induces the identity $A^n_T \to A^n_T \cong A^0(G_B) \otimes A^n_T$. Together with the first part of this proposition, it implies that $[(N, \mathrm{id})_*\mathbb{K}_{\underline{d}',\underline{d}''}(E_o+B)] = c_{1,\mathbb{K}} \in A^n\big(V_{\underline{d}',\underline{d}''}(\tilde{E}_o+B)\big)$. Using the same notations as in the proof of theorem 7.1, we deduce that for $a =$

$\chi_1^{m_1} \cdot \ldots \cdot \chi_r^{m_r}$ holds:

$$
\begin{aligned}
I^{1,(g-1)}_{\underline{d}',\underline{d}''}(a) &= \int_{\mathcal{V}_{\underline{d}',\underline{d}''}(E_o+B)} \Phi_T^*(a)/[s_0] = \sum_{K\in\mathcal{K}} \int_{K_{\underline{d}',\underline{d}''}(E_o+B)} \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r} \\
&= \sum_{K\in\mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K}\hookrightarrow\nu^{-1}K} \int_{\mathbb{K}_{\underline{d}',\underline{d}''}(E_o+B)} (N,\mathrm{id})^*\big(\Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r}\big) \\
&= \sum_{K\in\mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K}\hookrightarrow\nu^{-1}K} \int_{V_{\underline{d}',\underline{d}''}(\tilde{E}_o+B)} [(N,\mathrm{id})_*\mathbb{K}_{\underline{d}',\underline{d}''}(E_o+B)] \cdot \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r} \\
&= \sum_{K\in\mathcal{K}} \frac{1}{\nu_K} \sum_{\mathbb{K}\hookrightarrow\nu^{-1}K} \int_{V_{\underline{d}',\underline{d}''}(\tilde{E}_o+B)} c_{1,\mathbb{K}} \cdot \Lambda_1^{m_1} \cdot \ldots \cdot \Lambda_r^{m_r} \\
&\overset{(7.5)}{=} I^{0,(g-1)}_{\underline{d}',\underline{d}''}(c_1 \cdot a) \overset{\text{step 1}}{=} I^{g-1}_{\underline{d}}(c_1 \cdot a).
\end{aligned}
$$

It follows that $I^{1,(g-1)}_{\underline{d}',\underline{d}''}(a) = I^{g-1}_{\underline{d}}(c_1 \cdot a)$ for all $a \in A_T^{|\underline{d}|-n(g-1)}$. □

Step 3. the conclusion. Putting the previous two steps together, we find the main result of the paper:

Theorem 8.1. *The Hamiltonian Gromov-Witten invariants of the toric variety X fulfill the equality*

$$I^g_{\underline{d}}(a) = I^0_{\underline{d}}(c^g \cdot a), \quad \forall\, a \in A_T^{|\underline{d}|-n(g-1)},$$

with $c \in A_T^n$.

How to compute c ? This result would be rather abstract if we were unable to compute the class $c \in A_n^T$. We distinguish between two methods of computing it, but we must say that we are unable to write down a compact formula in general.

Method I This method is based on the localization technique described in section 6. In genus $g = 1$, the recursive formula reads $I^1_{\underline{d}}(a) = I^0_{\underline{d}}(c \cdot a)$, for all $a \in A_T^*$, and this condition determines c uniquely.

Method II Here we notice that, according to proposition 7.1 (iv), $N_* V_{\underline{d}}(C_o) \hookrightarrow V_{\underline{d}}(\tilde{C}_o)$ always contributes to c. This irreducible component is defined by the condition that its image through the evaluation map (7.2) is the closure of the $T \times T$-orbit of the diagonal in $\mathbb{C}^r \times \mathbb{C}^r$; we denote this closure by $\mathbb{D}$. The cycle $\mathbb{D}$ depends only on the 1-skeleton of the fan Σ, and only the other components of $\mathcal{V}_{\underline{d}}(C_o)$ probe deeper into the structure of Σ.

Here we limit ourselves to the computation of the $T \times T$-equivariant class $[\mathbb{D}] \in A^*_{T\times T}(\mathbb{C}^r \times \mathbb{C}^r)$. The trouble that one notices at the first glance is that $\mathbb{D}$ is a very singular variety, and therefore a pure force computation of its class is basically hopeless. We go round the problem as follows: we let $\iota : \Gamma \hookrightarrow \mathbb{C}^r \times X$ to be the closure of the graph of the quotient $\mathbb{C}^r \dashrightarrow X$; Γ is a toric variety for $(\mathbb{C}^*)^r$, and the inclusion ι is $(\mathbb{C}^*)^r$-equivariant. Now we notice that $\mathbb{D}$ fits into the diagram

$$\begin{tikzcd} \mathbb{D} \arrow[d, hook] & \Gamma \times_X \Gamma \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook] & \Delta_X \arrow[d, hook] \\ \mathbb{C}^r \times \mathbb{C}^r & \Gamma \times \Gamma \arrow[l, "(\mathrm{pr}_1, \mathrm{pr}_1)"'] \arrow[r, "(\mathrm{pr}_2, \mathrm{pr}_2)"] & X \times X, \end{tikzcd} \tag{8.2}$$

and $[\mathbb{D}] = (\mathrm{pr}_1)_*[\Gamma \times_X \Gamma]$.

We wish to use localization with respect to the $T \times T$-action, as described in [10], in order to compute the class $[\Gamma \times_X \Gamma] \in A_*^{T\times T}(\Gamma \times \Gamma)$. The first question which appears is what the corresponding fixed locus looks like? The answer is especially nice.

Lemma 8.1. $(\Gamma \times_X \Gamma)^{T\times T} = \{(0,0)\} \times X \hookrightarrow (\mathbb{C}^r \times X) \times_X (\mathbb{C}^r \times X)$.

Proof. Since X is projective, the cone generated by the characters $\{\chi_\rho\}_{\rho\in\Sigma(1)}$ form a strictly convex cone in $A^1(X)$, and we find a one parameter subgroup $\alpha : \mathbb{C}^* \to T$ such that $\langle \chi_\rho, \alpha\rangle > 0$ for all ρ. Then $\{0\} \subseteq (\mathbb{C}^r)^T \subseteq (\mathbb{C}^r)^{\alpha(\mathbb{C}^*)} = \{0\}$. □

The trouble is that Γ is typically singular and, though localization is still valid, the *localization formula* is not explicit. Therefore we will localise on $(\mathbb{C}^r \times X) \times_X (\mathbb{C}^r \times X)$, and intersect the resulting class with $[\Gamma \times \Gamma]$. For shorthand, we will write $Y := \mathbb{C}^r \times X$. Since for regular toric varieties the cycle map $A_*(X) \to H_*(X;\mathbb{Z})$ is an isomorphism, the equality

$$[\Delta_X] = \sum_\kappa h'_\kappa \otimes h''_\kappa \tag{8.3}$$

holds in $A_*(X) \otimes A_*(X)$, for $\{h'_\kappa\}_\kappa$ and $\{h''_\kappa\}_\kappa$ dual bases in $A_*(X)$. By taking the flat pull-back by $(\mathrm{pr}'_2, \mathrm{pr}''_2) : Y \times Y \to X \times X$, we obtain

$$(\mathrm{pr}'_2, \mathrm{pr}''_2)^*[\Delta_X] = \sum_\kappa (\mathrm{pr}'_2)^* h'_\kappa \otimes (\mathrm{pr}''_2)^* h''_\kappa \in A_*^T(Y) \otimes A_*^T(Y).$$

The inclusion $\iota : \Gamma \hookrightarrow Y$ is $(\mathbb{C}^*)^r$-invariant, and defines the equivariant class $[\Gamma] \in A_*^T(Y)$, and our previous discussion shows that

$$[\mathbb{D}] = (\mathrm{pr}_1)_*\big((\mathrm{pr}'_2, \mathrm{pr}''_2)^*[\Delta_X] \cdot [\Gamma \times \Gamma]\big).$$

Using the equality (8.3), we are reduced to compute intersection products of type

$$a = \mathrm{pr}_2^* h \cdot [\Gamma] \in A_*^T(Y), \quad h \in A_*(X).$$

If $\iota_X : X = \{0\} \times X \hookrightarrow \mathbb{C}^r \times X$ denotes the inclusion, the localization formula for the class a reads

$$a = (\iota_X)_* \frac{\iota_X^* a}{\mathsf{e}^T(\mathcal{N}_{X|Y})} \in A_*^T(Y).$$

The equivariant Euler class of the normal bundle $\mathcal{N}_{X|Y} \cong \mathcal{O}_X^{\oplus r}$ is

$$\mathsf{e}^T(\mathcal{N}_{X|Y}) = \textstyle\prod_\rho \chi_\rho \in A_T^*,$$

and, according to [12, corollary 6.3 and corollary 8.1.1],

$$\iota_X^* a = a \cdot [X] = (\mathrm{pr}_2^* h \cdot [\Gamma]) \cdot [X] = (\mathrm{pr}_2^* h \cdot [X]) \cdot [\Gamma] = h \cdot \mathsf{e}^T(\mathcal{N}_{\Gamma|Y}).$$

9. Examples

In this section we want to present a few concrete computations of these Hamiltonian invariants, which should illustrate the general theory presented in this paper.

(I) $X = \mathbb{P}^{r-1}$ This example is classical and well understood. For $d > 2g-1$, the compactification of $\mathsf{Mor}_d(C, \mathbb{P}^{r-1})$ is

$$V_d(C) = \mathbb{P}(\mathcal{W}^{\oplus r}) \longrightarrow \mathcal{J},$$

for $\mathcal{W} \to \mathcal{J}$ the Picard bundle. In this case there is only one 'natural' line bundle $\Lambda \to V_d(C)$ over this moduli space, and the corresponding invariant is

$$I_d^g = \int_{V_d(C)} \Lambda^{rd-(r-1)(g-1)},$$

which equals r^g. The genus recursion formula in theorem 7.1 boils down to $I_d^g = r \cdot I_d^{g-1}$, with $r =$ Euler number of $\mathbb{P}^{r-1}$, and the class $c = r \cdot \chi^{r-1}$. We can also see that the enumerative meaning of the Hamiltonian invariants is not always obvious: the value of the genus $g = 1$ and degree $d = 1$ invariant equals r, though there are no morphisms of degree one from an elliptic curve to $\mathbb{P}^{r-1}$.

(II) $X = \mathbb{F}_l$ is the l^{th} Hirzebruch surface. The torus $T = \mathbb{C}_t^* \times \mathbb{C}_\tau^*$ acts on $\mathbb{C}^4$ by

$$1 \longrightarrow T \xrightarrow{\varepsilon} (\mathbb{C}^*)^4, \quad (t, \tau) \mapsto (t, \tau, t, t^l\tau),$$

and $\mathbb{F}_l = \Omega/T$ with $\Omega := \mathbb{C}^4 \setminus \big\{\{z_1 = z_3 = 0\} \cup \{z_2 = z_4 = 0\}\big\}$. It carries the invariant divisors $D_1, \ldots, D_4$ which satisfy $[D_1] = [D_3],\ [D_4] = [D_2] + l[D_1],\ D_1^2 = D_3^2 = 0,\ D_2^2 = -l,\ D_4^2 = l$. Alternatively, $\mathbb{F}_l = \mathbb{P}\big(\mathcal{O} \oplus \mathcal{O}(l)\big) \xrightarrow{a} \mathbb{P}^1$, and the line bundles defined by the invariant divisors are as follows:

$$\mathcal{O}(D_1) \cong \mathcal{O}(D_3) \cong a^*\mathcal{O}_{\mathbb{P}^1}(1), \quad \mathcal{O}(D_2) = \mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(l))}(1).$$

Let C be a smooth curve of genus g and let $\mathcal{J}$ be its Jacobian variety. For two integers $d, d' > 2g-1$, we are going to describe the compactification of the space of morphisms from C into $\mathbb{F}_l$ which have multi-degree $\underline{d} = (d, d', d, ld + d')$. The morphism (2.12) takes in this case the form

$$\mathcal{J}^2 \longrightarrow \mathcal{J}^4, \quad (x, y) \longmapsto \big(x, y, x, \psi(x, y) := lx + y\big),$$

and we get the diagram

$$\begin{tikzcd}
\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_4 \arrow[d] & \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_4 =: \mathcal{V} \arrow[d] \\
\mathcal{J}^2 \times C \arrow[r, "\mathrm{pr}_1"] & \mathcal{J}^2,
\end{tikzcd}$$

where $\mathcal{L}_4 \to \mathcal{J}^2$ is the pull-back by $\psi : \mathcal{J}^2 \to \mathcal{J}$ of the Poincaré sheaf of degree $ld + d'$. The 'nice open set'

$$\mathcal{V}^o = \mathcal{V} \setminus \big\{\{0 \oplus \mathcal{V}_2 \oplus 0 \oplus \mathcal{V}_4\} \cup \{\mathcal{V}_1 \oplus 0 \oplus \mathcal{V}_3 \oplus 0\}\big\}$$

and T acts by ε on it, with quotient $V_{\underline{d}}(C)$. By taking the quotient in two stages, first for the $\mathbb{C}^*_t$-action and then for the remaining $\mathbb{C}^*_\tau$-action, we see that $V_{\underline{d}}(C)$ fits into the diagram

$$\begin{tikzcd}
V_{\underline{d}}(C) = \mathbb{P}\big(b^*\mathcal{V}_2 \oplus \Lambda_1^l \otimes b^*\mathcal{V}_4\big) \arrow[r, "p"] \arrow[dr, "q"'] & \mathbb{P}(\mathcal{V}_1 \oplus \mathcal{V}_3) \times \mathcal{J}_{d'} \arrow[d, "b"] \\
 & \mathcal{J}_d \times \mathcal{J}_{d'},
\end{tikzcd}$$

where $\Lambda_1 := \mathcal{O}_{\mathbb{P}(\mathcal{V}_1 \oplus \mathcal{V}_3)}(1)$. We put $\Lambda_2 := \mathcal{O}_{\mathbb{P}(b^*\mathcal{V}_2 \oplus \Lambda_1^l \otimes b^*\mathcal{V}_4)}(1)$ and $\Lambda_4 = \Lambda_1^l \otimes \Lambda_2$. The Euler sequence on $V_{\underline{d}}(C)$ is

$$0 \longrightarrow \Lambda_4^{-1} \longrightarrow p^*\big(\Lambda_1^{-l} \otimes b^*\mathcal{V}_2 \oplus b^*\mathcal{V}_4\big) \longrightarrow \Lambda_4^{-1} \otimes \mathsf{T}_p^{\mathrm{rel}}$$

and it follows that

$$p_*\big[(1 - \Lambda_4)^{-1}\big] \cdot c\big(\Lambda_1^{-l} \otimes b^*\mathcal{V}_2 \oplus b^*\mathcal{V}_4\big) = 1,$$

and therefore

$$\begin{aligned}
p_*\big[(1 - \Lambda_4)^{-1}\big] &= c\big(\Lambda_1^{-l} \otimes b^*\mathcal{V}_2 \oplus b^*\mathcal{V}_4\big)^{-1} = c\big(\Lambda_1^{-l} \otimes b^*\mathcal{V}_2\big)^{-1} \cdot c(\mathcal{V}_4)^{-1} \\
&= \exp\left(\frac{\theta_2}{1 - l\Lambda_1}\right) \cdot (1 - l\Lambda_1)^{-N_2} \cdot \exp(\psi^*\theta_4).
\end{aligned}$$

The expended form of the first two factors is

$$\begin{aligned}
&\exp\left(\frac{\theta_2}{1 - l\Lambda_1}\right) \cdot (1 - l\Lambda_1)^{-N_2} \\
&\quad = \exp(\theta_2) \cdot \big[1 + l(\theta_2 + N_2)\Lambda_1 + l^2\big(\theta_2^2 + 2(N_2+1)\theta_2 + N_2(N_2 + 1)\big)\Lambda_1^2 + \ldots\big],
\end{aligned}$$

while the third factor is

$$\psi^*\theta_4 = l^2\theta_1 + \theta_2 + l\mu, \qquad \text{with } \mu \in H^1(\mathcal{J}_d) \otimes H^1(\mathcal{J}_{d'}).$$

Some computations show that

$$\frac{\theta_1^{g-1}}{(g-1)!} \cdot \frac{\theta_2^{g-1}}{(g-1)!} \cdot \frac{\mu^2}{2} = -g \cdot \frac{\theta_1^g}{g!} \cdot \frac{\theta_2^g}{g!}.$$

With these preparations, we can finally compute some Hamiltonian invariants of the Hirzebruch surface $\mathbb{F}_l$: these are integrals

$$I^g_{\underline{d}}(m_1, m_4) = \int_{V_{\underline{d}}(C)} \Lambda_1^{m_1} \Lambda_4^{m_4} = \int_{\mathbb{P}(\mathcal{V}_1 \oplus \mathcal{V}_3) \times \mathfrak{J}_{d'}} \Lambda_1^{m_1} \cdot p_* \Lambda_4^{m_4}$$

with $m_1 + m_4 = \dim V_{\underline{d}}(C)$, all the others being linear combinations of such.

(1) For $m_1 > N_1 + N_3 + g - 1 = \dim \mathbb{P}(\mathcal{V}_1 \oplus \mathcal{V}_3)$, $I^g_{\underline{d}}(m_1, m_4) = 0$.
(2) For $m_1 = N_1 + N_3 + g - 1$ and $m_4 = N_2 + N_4 + g - 1$,

$$\begin{aligned} I^g_{\underline{d}}(m_1, m_4) &= \int_{\mathbb{P}(\mathcal{V}_1 \oplus \mathcal{V}_3) \times \mathfrak{J}_{d'}} \Lambda_1^{N_1+N_3+g-1} \exp(\theta_2) \exp(\psi^* \theta_4) \\ &\qquad \cdot \left[1 + l(\theta_2 + N_2)\Lambda_1 + \dots\right] \\ &= 2^g \int_{\mathfrak{J}_{d'}} \exp(\theta_2) \cdot \exp(\theta_2) = 2^g \cdot 2^g = 4^g. \end{aligned}$$

(3) For $m_1 = N_1 + N_3 + g - 2$ and $m_4 = N_2 + N_4 + g$ some calculations show that

$$I^g_{\underline{d}}(m_1, m_4) = l^2 g 4^{g-1} + l\left(d' - \frac{g}{2} + 1\right) 4^g.$$

By increasing the value of m_4 the computations become more lengthy and tedious. However these values are sufficient for checking the equality (5.3), and the genus recursion formula in theorem 7.1.

Let e, e' be positive integers, and consider the multi-degree $\underline{\tilde{d}} := (d + e, d' + e', d + e, l(d + e) + d' + e')$: then (5.3) says that

$$\int_{V_{\underline{\tilde{d}}}(C)} \Lambda_1^{m_1+e} \Lambda_2^{e'} \Lambda_3^{e} \Lambda_4^{m_4+le+e'} = \int_{V_{\underline{d}}(C)} \Lambda_1^{m_1} \Lambda_4^{m_4}$$

should hold. We check this for m_1, m_4 as in (3) above; then the left hand side is

$$\begin{aligned} &\int_{V_{\underline{\tilde{d}}}(C)} \Lambda_1^{m_1+2e} \Lambda_2^{e'} \Lambda_4^{m_4+le+e'} \\ &= \int_{V_{\underline{\tilde{d}}}(C)} \Lambda_1^{\tilde{N}_1+\tilde{N}_3+g-2} (\Lambda_4 - l\Lambda_1)^{e'} \Lambda_4^{\tilde{N}_2+\tilde{N}_4+g-e'} \\ &= \int_{V_{\underline{\tilde{d}}}(C)} \Lambda_1^{\tilde{N}_1+\tilde{N}_3+g-2} \Lambda_4^{\tilde{N}_2+\tilde{N}_4+g} - le' \cdot \int_{V_{\underline{\tilde{d}}}(C)} \Lambda_1^{\tilde{N}_1+\tilde{N}_3+g-1} \Lambda_4^{\tilde{N}_2+\tilde{N}_4+g-1} \\ &= l^2 g 4^{g-1} + l\left((d' + e') - \frac{g}{2} + 1\right) 4^g - le' 4^g = l^2 g 4^{g-1} + l\left(d' - \frac{g}{2} + 1\right) 4^g. \end{aligned}$$

So the numerical computation agrees with the theoretical prediction, as it should.

We are going to check now the genus recursion formula 7.1. The class of the diagonal in $\mathbb{F}_l \times \mathbb{F}_l$ is

$$\Delta = [\mathbb{F}_l] \otimes [*] + [D_1] \otimes [D_2] + [D_4] \otimes [D_1] + [*] \otimes [\mathbb{F}_l] \in A_*(\mathbb{F}_l) \otimes A_*(\mathbb{F}_l).$$

Let $\chi_1,\dots,\chi_4$ be the characters of the T-action on $\mathbb{C}^4$. Using localization, we find that the class $\mathbb{D}$ appearing in (8.2) equals

$$\mathbb{D} = 1\otimes\chi_1\chi_4 + \chi_1\otimes\chi_2 + \chi_4\otimes\chi_1 + \chi_1\chi_4\otimes 1 \in A_*^T\otimes A_*^T.$$

An alternative way for computing this is as follows: we notice that the ideal defining $\mathbb{D}\hookrightarrow\mathbb{C}^4\times\mathbb{C}^4$ is $(z_1w_3 - z_3w_1, z_1^l z_2w_4 - z_4w_1^l w_2, z_2z_3^l w_4 - z_4w_2w_3^l)$. The scheme corresponding to the ideal generated by the first two polynomials consists of the union of $\mathbb{D}$ with $\{z_1 = w_1 = 0\}$, and this coordinate plane has multiplicity l in the scheme. Therefore the equivariant class is

$$\big[(\chi_1\otimes 1 + 1\otimes\chi_1)\otimes(\chi_4\otimes 1 + 1\otimes\chi_4)\big] - l\chi_1\otimes\chi_1,$$

as stated. For m_1, m_4 be as in (3) above, we find that

$$\begin{aligned}
&I_{\underline{d}}^{g-1}(\mathbb{D}, m_1, m_4)\\
&= \int_{V_{\underline{d}}(\tilde{C})}\big(\Lambda_1\Lambda_4 + \Lambda_1\Lambda_2 + \Lambda_1\Lambda_4 + \Lambda_1\Lambda_4\big)\Lambda_1^{N_1+N_3+g-2}\Lambda_4^{N_2+N_4+g}\\
&= 4\int_{V_{\underline{d}}(\tilde{C})}\Lambda_1^{\tilde{N}_1+\tilde{N}_3+(g-1)-2}\Lambda_4^{\tilde{N}_2+\tilde{N}_4+(g-1)}\\
&\quad -l\int_{V_{\underline{d}}(\tilde{C})}\Lambda_1^{\tilde{N}_1+\tilde{N}_3+(g-1)-1}\Lambda_4^{\tilde{N}_2+\tilde{N}_4+(g-1)-1}\\
&= l^2(g-1)4^{g-1} + l\left(d' - \frac{g-1}{2} + 1\right)4^g - l4^{g-1}.
\end{aligned}$$

The difference

$$I_{\underline{d}}^{g}(m_1, m_4) - I_{\underline{d}}^{g-1}(\mathbb{D}, m_1, m_4) = l(l-1)4^{g-1}$$

counts for the contribution of the 'unpleasant component' which appears in the degeneration process. We can describe precisely what is going on in this case: for the primitive collection $\pi = \{1,3\}$, the multi-degree $\underline{d}_{\{1,3\}} = (1,0,1,l)$ corresponds to the class $[D_4]\in A_1(\mathbb{F}_l)$. We notice that any two points on $\mathbb{F}_l$ can be joined by a (possibly reducible) curve whose class is $[D_4]$. Lemma 7.3 implies that the component $K_{\{1,3\}}(C_o)$ represents a multiple of the class $\Lambda_1\Lambda_3 = \Lambda_1^2\in A_*(V_{\underline{d}}(\tilde{C}_o))$, the multiplicity coming from the fact that the corresponding scheme structure is non-reduced. Taking a glance at the previous equality, we see that the multiplicity of $K_{\{1,3\}}(C_o)$ equals $l(l-1)$.

Further, let us notice that, from dimensional counting reasons, the another primitive collection $\{2,4\}$ contributes trivially to the invariant. The final result is that, for the Hirzebruch surface $\mathbb{F}_l$, the Hamiltonian invariant obeys the recursion formula

$$\begin{aligned}
I_{\underline{d}}^{g}(a) = I_{\underline{d}}^{g-1}\Big(\big(\mathbb{D} + l(l-1)\chi_1^2\big)\cdot a\Big) &= I_{\underline{d}}^{0}\Big(\big(\mathbb{D} + l(l-1)\chi_1^2\big)^g\cdot a\Big).\\
&= I_{\underline{d}}^{0}\Big(\big(4x_1x_4 + l^2x_1^2\big)^g\cdot a\Big).
\end{aligned}$$

A. Some results about toric varieties

The goal of this appendix is to recall some facts concerning toric varieties which are needed in the paper. The notations that we are using are those of section 2.

1. Toric varieties as invariant quotients We wish to mention a probably well-known generality about toric varieties: by the work of D. Cox, we know that toric varieties are categorical quotients of open subsets of affine spaces. What we are going to prove here is that, for suitable linearizations, they are actually GIT quotients (in the sense of D. Mumford).

Proposition A.1. *Let X be a (not necessarily smooth) projective toric variety, and let $\beta \in A^1(X) = \mathcal{X}^*(T)$ be an ample class. We linearize the T-action in the trivial line bundle $A \to \mathbb{C}^r$ as follows:*

$$t \times (z, w) := (\varepsilon(t) \times z, \beta(t) \cdot w), \quad \forall t \in T \text{ and } \forall (z, w) \in A = \mathbb{C}^r \times \mathbb{C},$$

where the monomorphism $\varepsilon : (\mathbb{C}^*)^l \to (\mathbb{C}^*)^r$ has been defined in (2.7). *Then the following hold:*

(i) *the linearization above can be extended to a linearization of the standard $(\mathbb{C}^*)^r$-action on $\mathbb{C}^r$;*
(ii) *the T-semi-stable locus for this linearization is precisely $\Omega \subset \mathbb{C}^r$;*
(iii) *$\mathbb{C}^r /\!/_\beta T = X$, and the induced polarisation on X is β.*

Proof. Let us denote

$$S_\beta := \left\{ f \in \mathbb{C}[X_1, \ldots, X_r] \mid f(\varepsilon(t) \times z) = \beta(t) \cdot f(z), \ \forall t \in T, \ \forall z \in \mathbb{C}^r \right\}.$$

Since the T- and the $(\mathbb{C}^*)^r$-actions on $\mathbb{C}^r$ commute, $(\mathbb{C}^*)^r$ leaves S_β invariant. Consequently, S_β decomposes into weight spaces and we choose $\mu : (\mathbb{C}^*)^r \to \mathbb{C}^*$ one of these weights. Then $\mu \circ \varepsilon = \beta$ and we can linearise the $(\mathbb{C}^*)^r$-action in $A = \mathbb{C}^r \times \mathbb{C}$ by

$$t' \times (z, w) := (t' \times z, \mu(t') \cdot w), \quad \forall t' \in (\mathbb{C}^*)^r.$$

This proves our first statement.

It is proved in [6, proposition 1.1], that

$$H^0(X, \mathcal{O}_X(\beta)) \cong S_\beta.$$

$\mathcal{O}_X(\beta) \to X$ being ample, it is globally generated; since $X = \Omega/T$, for any $z \in \Omega$ we find a polynomial $f \in S_\beta$ such that $f(z) \neq 0$. On the other hand, S_β can be naturally identified with the vector space of β-equivariant sections $\mathbb{C}^r \to A$. By the very definition of the semi-stability, we deduce that $\Omega \subset \mathbb{C}^r_{\beta-\mathrm{ss}}$.

For proving that Ω coincides with the β-semi-stable locus, it remains to prove that the restriction of any polynomial in S_β to $Z_X = \mathbb{C}^r \setminus \Omega$ is identically zero.

Let us consider $f = \prod_{\rho=1}^r X_\rho^{k_\rho} \in S_\beta$ a monomial and $\pi \subset \Sigma(1)$ a primitive family. We must prove that not all $\{k_\rho\}_{\rho \in \pi}$ vanish.

Let us assume the contrary, that $k_\rho = 0$ for all $\rho \in \pi$. Since β is ample, there exists a strictly convex, Σ-linear function $\eta : \mathbb{R}^r \to \mathbb{R}$ such that $\eta(e^\rho) = k_\rho$ for all $\rho = 1, \dots, r$. In particular, as η is piecewise linear, we deduce that $\eta \geq 0$. On the other hand, according to [2, theorem 4.6], we have the following inequality

$$\eta\Big(\sum_{\rho\in\pi} e^\rho\Big) < \sum_{\rho\in\pi} \eta(e^\rho) = \sum_{\rho\in\pi} k_\rho = 0.$$

This is a contradiction which proves that $\Omega = \mathbb{C}^r_{\beta-\mathrm{ss}}$.

We turn now to the third part of the proposition. Since, on one hand $X = \Omega/T$ is a categorical quotient (according to [6], theorem 2.1) and, on the other hand, $\mathbb{C}^r /\!\!/_\beta T = \Omega/T$ is again a categorical quotient, we deduce from the unicity of a categorical quotient that $X = \mathbb{C}^r /\!\!/_\beta T$ as a variety. It remains to look at the induced polarisation on X. Choose a writing $\sum_{\rho=1}^r k_\rho D_\rho$ for β: then, for any cone $\sigma \in \Sigma$ we obtain an isomorphism of modules

$$H^0(\Omega_\sigma, A)^\beta \longrightarrow H^0(X_\sigma, \mathcal{O}_X(\beta)),$$

$$f \longmapsto f \cdot \Bigg[\prod_{\rho=1}^r X_\rho^{k_\rho}\Bigg]^{-1}.$$

In the formulae above we denoted $\Omega_\sigma := \{z_\rho \neq 0 \mid \forall \rho \notin \sigma(1)\}$ and $X_\sigma := \Omega_\sigma/T$; these are the 'affine building blocks' for X. It follows from these isomorphisms that the induced line bundle on X is $\mathcal{O}_X(\beta)$, and therefore the induced polarisation on X is the one defined by β. □

Corollary A.1. *In the context of the previous proposition, assume moreover that X is smooth. Then*

(i) Ω *coincides with the properly T-stable locus;*
(ii) $X = \Omega/T$ *is a geometric quotient.*

Proof. The statements follow from the fact that when X is smooth, the action of T on Ω is free and is also closed (see the proof of theorem 2.1 in [6]). □

2. The ample cone and the cone of effective curves of a toric variety In this subsection we further assume that Σ is regular: then we have the two basic short exact sequences

$$0 \longrightarrow K \longrightarrow \oplus_\rho \mathbb{Z}w_\rho \xrightarrow{a} N \longrightarrow 0, \quad w_\rho \overset{a}{\longmapsto} e^\rho,$$

and

$$0 \longrightarrow M \longrightarrow \oplus_\rho \mathbb{Z}w_\rho^\vee \longrightarrow A^1(X) \longrightarrow 0, \quad w_\rho^\vee \overset{c}{\longmapsto} c(w_\rho^\vee) = [D_\rho].$$

Let us notice that the elements of K are multi-degrees $\underline{d} = (d_\rho)_\rho$ with the property that $d_\rho e^\rho = 0$.

Following [20], we define

$$\mathrm{CPL}(X) := \left\{ \eta : N_{\mathbb{R}} \to \mathbb{R} \;\middle|\; \begin{array}{l} \eta \text{ is } \Sigma\text{-linear, and} \\ \eta(w' + w'') \leq \eta(w') + \eta(w''),\ \forall w', w'' \in N_{\mathbb{R}} \end{array} \right\}$$

which is called the cone of Σ-linear, convex functions on $N_{\mathbb{R}}$, and

$$\mathrm{cpl}(X) := \left\{ \alpha(\eta) := \sum_{\rho} \eta(e^{\rho})[D_{\rho}] \;\middle|\; \eta \in \mathrm{CPL}(X) \right\} \subset A^1(X)_{\mathbb{R}}.$$

which is the cone of nef divisors on X. Both cones are strictly convex and polyhedral, and the interior of $\mathrm{cpl}(X)$ corresponds to the Kähler classes on X. A point lies in the interior of $\mathrm{cpl}(X)$ if there is a strictly convex, Σ-linear function on $N_{\mathbb{R}}$ defining it.

Proposition A.2. *[2, theorem 4.6] A Σ-linear function $\eta : N_{\mathbb{R}} \to \mathbb{R}$ is strictly convex if and only if*

$$\sum_{\rho \in \pi} \eta(e^{\rho}) > \eta\left(\sum_{\rho \in \pi} e^{\rho}\right),$$

for all primitive families $\pi \subset \Sigma(1)$.

As an immediate consequence of this result we obtain the description of the facets of $\mathrm{CPL}(X)$.

Remark A.1. The facets (i.e. the codimension one faces) of $\mathrm{CPL}(X)$ are of the form

$$F_{\pi} := \left\{ \eta \in \mathrm{CPL}(X) \;\middle|\; \sum_{\rho \in \pi} \eta(e^{\rho}) = \eta\left(\sum_{\rho \in \pi} e^{\rho}\right) \right\}, \text{ with } \pi \text{ a primitive family.}$$

Obviously, the facets of $\mathrm{CPL}(X)$ define the facets of $\mathrm{cpl}(X)$ and conversely.

There is yet another interesting object associated to X, namely the Mori cone; it is traditionally denoted by $\mathrm{NE}(X)$ and its points parameterize linear combinations, with positive coefficients, of effective 1-cycles modulo numerical equivalence. A detailed presentation of it can be found in [19, section 2.5]. By very definition, $\mathrm{NE}(X)$ is a strictly convex cone which is contained in $K_{\mathbb{R}}$.

Theorem A.1. *[20, theorem 2.3, (2)]*

$$\begin{aligned} \mathrm{NE}(X) &= \mathrm{CPL}(X)^{\vee} \\ &= \left\{ \underline{d} = (d_{\rho})_{\rho} \in K_{\mathbb{R}} \;\middle|\; \langle \eta, \underline{d} \rangle = \sum_{\rho} d_{\rho}\eta(e^{\rho}) \geq 0, \forall \eta \in \mathrm{CPL}(X) \right\}. \end{aligned}$$

Remark A.2. (i) From this duality it follows that the extremal rays of $\mathrm{NE}(X)$ bijectively correspond to the facets of $\mathrm{CPL}(X)$ in the following way: $R \subset \mathrm{NE}(X)$ is an extremal ray if and only if there is a primitive family π such that

$$R = \left\{ \underline{d} \in \mathrm{NE}(X) \mid \langle \eta, \underline{d} \rangle = 0,\ \forall \eta \in F_{\pi} \right\} = \mathbb{Q}_{+} R_{\pi},$$

where R_{π} is the so-called relation of π (see [3, definition 2.8]).

(ii) For a primitive collection π, we define an effective class as follows: since the fan Σ is complete, there is a unique cone $\sigma(\pi)$ which contains $-\sum_{\rho\in\pi} e^\rho$ in its relative interior (notice the '$-$' sign, opposite to [3, definition 2.7]). Then there are uniquely defined, strictly positive integers $(l_\rho)_{\rho\in\sigma(\pi)}$ such that

$$\sum_{\rho\in\pi} e^\rho + \sum_{\rho\in\sigma(\pi)} l_\rho e^\rho = 0.$$

Let us notice that for this choice it might happen that $\pi\cap\sigma(\pi)\neq\emptyset$. We define now the class $\underline{d}_\pi \in \mathrm{NE}(X)$ by

$$d_{\pi,\rho} := \begin{cases} 1 \ \text{ if } \rho\in\pi\setminus\sigma(\pi), \\ 1+l_\rho \ \text{ if } \rho\in\pi\cap\sigma(\pi), \\ l_\rho \ \text{ if } \rho\in\sigma(\pi)\setminus\pi, \\ 0 \ \text{ else.} \end{cases}$$

We remark that $\underline{d}_\pi$ does not define an extremal ray unless $\sum_{\rho\in\pi} e^\rho = 0$ (according to [3, theorem 2.15]). It is proved in [3, proposition 3.2] that such collections exist for any complete and regular fans.

B. Remarks about the compactified Jacobian

Here we wish to recall a few facts related to the Jacobian (Picard) variety of a smooth and projective curve, and also about the compactified Jacobian of a singular, nodal curve. The question of compactifying the Picard variety of singular curves, or families of reduced and connected curves, raised quite a lot of interest since Igusa's paper [16] and required many efforts to be answered.

In our paper we need to have a good grip of the compactified Jacobian only in the very simple cases when a smooth and irreducible curve degenerates to an irreducible curve with one node and into a curve with two smooth components. Though from a specialist's point of view these cases are classical, our wish is to give a down-to-earth description of the compactified Jacobian, good enough to permit us describing the degeneration of our moduli space $\mathrm{Mor}_{\underline{d}}(C, X)$ and of its compactification.

Part I: *degeneration to a curve with one node* Our presentation will follow very closely that of Igusa, which we prefer for its explicit character. Let C be a smooth and irreducible, projective curve of genus g and $d > 2g-2$ an integer. Then it is well known that the d^{th} symmetric power $C(d)$ of C fibres over the Jacobian

$$C(d) \longrightarrow \mathfrak{J}_d, \quad \underline{x} = x_1 + \ldots + x_d \longmapsto [\mathcal{O}_C(\underline{x})].$$

The fibre over $[L] \in \mathfrak{J}_d$ is the linear system $|L|$, so that we can say that $\mathfrak{J}_d$ parameterises the divisors of degree d on C modulo rational equivalence. Equivalently, $\mathfrak{J}_d$ parameterizes isomorphism classes of line bundles of degree d over C.

Let us consider now an irreducible, projective curve C_0 with exactly one node, and let $\tilde{C}_0 \xrightarrow{n} C_0$ be its normalisation. We denote by z the node of C_0 and by $z', z'' \in \tilde{C}_0$ the points lying above it.

A line bundle on C_0 corresponds to a line bundle on $\tilde{C}_0$ and the choice of an isomorphism between the stalks at z' and z''. Consequently we find the exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathsf{Pic}^d(C_0) \xrightarrow{n^*} \mathsf{Pic}^d(\tilde{C}_0) \longrightarrow 0,$$

and the question that one faces is: what kind of geometric objects on C_0 should be added to $\mathsf{Pic}^d(C_0)$ in order to compactify it? and what does this compactification looks like?

Let us consider the divisor $\underline{x} = x_1 + \ldots + x_d$ whose support consists of d distinct, smooth points of C_0, and let $\underline{\tilde{x}} := n^*\underline{x}$ be the corresponding divisor on $\tilde{C}_0$. Then clearly $n^*\mathcal{O}_{C_0}(\underline{x}) = \mathcal{O}_{\tilde{C}_0}(\underline{\tilde{x}})$. We shall keep the points $\tilde{x}_2, \ldots, \tilde{x}_d$ fixed and let $\tilde{x}_1$ approach z'. This is the same as approaching the node P with points $x_{1,\varepsilon}$ lying on one of the branches of C_0 at z. We wish to find the limit of the line bundles $\mathcal{O}_{C_0}(\underline{x}_\varepsilon)$, with $\underline{x}_\varepsilon := x_{1,\varepsilon} + x_2 + \ldots + x_d$, as $x_{1,\varepsilon} \to z$. By interpreting the sections in these line bundles as meromorphic functions on C_0 with at most simple poles located along the support of the corresponding divisors, we find that

$$n^*H^0\big(C_0, \mathcal{O}_{C_0}(\underline{x}_\varepsilon)\big) = \big\{f \in H^0\big(\tilde{C}_0, \mathcal{O}_{\tilde{C}_0}(\underline{\tilde{x}}_\varepsilon)\big) \mid f(z') = f(z'')\big\} \subset \mathcal{M}ero(\tilde{C}_0).$$

As $x_{1,\varepsilon} \to z$, its inverse image $\tilde{x}_{1,\varepsilon} \to z'$, and the limit of these vector spaces consists of meromorphic functions f on $\tilde{C}_0$ with at most simple poles at $z', x_2, \ldots, x_d \in C_0$, having the additional property that $f(z') = f(z'')$. Since f is regular at z'', we deduce that f must be regular at z' too. This means that f actually belongs to the linear system

$$|\tilde{x}_2 + \ldots + \tilde{x}_d| = |\lim_{\varepsilon \to 0} \underline{\tilde{x}}_\varepsilon - z'|,$$

and has the additional property that $f(z') = f(z'')$. The conclusion of our discussion is that $\lim_{\varepsilon\to 0} \mathcal{O}_{C_0}(\underline{x}_\varepsilon) = n_*\mathcal{O}_{\tilde{C}_0}(\lim_{\varepsilon\to 0} \underline{\tilde{x}}_\varepsilon - z')$, and in order to compactify $\mathsf{Pic}^d(C_0)$ we should add the isomorphism classes of the rank one, torsion free sheaves

$$n_*\big(\tilde{L}(-z')\big) \text{ and } n_*\big(\tilde{L}(-z'')\big), \text{ with } \tilde{L} \in \mathsf{Pic}^d(\tilde{C}_0) =: \tilde{\mathfrak{J}}_d.$$

With this picture in mind, we are going to give a concrete description of the compactified Jacobian of C_0, that we are going to denote still by $\mathfrak{J}_d$. We fix a *smooth point* $\zeta_0 \in C_0$, let $\tilde{\zeta}_0 \in \tilde{C}_0$ be the point above it, and consider the Poincaré bundle $\tilde{\mathcal{L}} \to \tilde{\mathfrak{J}}_d \times \tilde{C}_0$ trivialised at $\tilde{\zeta}_0$. We have two special points z' and z'' on $\tilde{C}_0$ which are identified by the normalisation map n. Let us define the family $\mathcal{M}$ of coherent, rank one, torsion free sheaves over C_0 by means of the following diagram

$$\begin{tikzcd}
 & \mathrm{pr}^*\tilde{\mathcal{L}} \arrow[d] & \tilde{\mathcal{L}} \arrow[d] \\
 & \mathbb{P}\big(\tilde{\mathcal{L}}_{z'} \oplus \tilde{\mathcal{L}}_{z''}\big) \times \tilde{C}_0 \arrow[r, "\mathrm{pr}"] \arrow[d, "(id,n)"] & \tilde{\mathfrak{J}}_d \times \tilde{C}_0 \\
\mathcal{M} := (id,n)_*\,\mathrm{pr}^*\tilde{\mathcal{L}} \arrow[r] & \mathbb{P}\big(\tilde{\mathcal{L}}_{z'} \oplus \tilde{\mathcal{L}}_{z''}\big) \times C_0 &
\end{tikzcd}$$

Over $\mathbb{J}_d := \mathbb{P}(\tilde{\mathcal{L}}_{z'} \oplus \tilde{\mathcal{L}}_{z''})$ we have the exact sequence

$$0 \longrightarrow \tau \longrightarrow \mathrm{pr}^*(\tilde{\mathcal{L}}_{z'} \oplus \tilde{\mathcal{L}}_{z''}) \xrightarrow{\varphi} \lambda \longrightarrow 0,$$

where τ is the tautological line bundle. We notice that in this case the quotient λ is still a line bundle. We denote by

$$\imath_0 : \mathbb{J}_d = \mathbb{J}_d \times \{z\} \hookrightarrow \mathbb{J}_d \times C_0$$

the inclusion. Finally we consider the homomorphism of sheaves h defined by

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathrm{ev}_z"] \arrow[rr, bend right, "h"'] & (\imath_0)_* \mathrm{pr}^*(\tilde{\mathcal{L}}_{z'} \oplus \tilde{\mathcal{L}}_{z''}) \arrow[r, "\varphi"] & (\imath_0)_*\lambda, \end{tikzcd} \tag{B.1}$$

and let $\mathbb{L}_d \to \mathbb{J}_d \times C_0$ to be its kernel.

Proposition B.1. (i) *$\mathbb{L}_d$ is trivialised along $\mathbb{J}_d \times \{\zeta_0\}$;*

(ii) *Denote by $\tilde{\mathfrak{J}}' := \mathbb{P}(\tilde{\mathcal{L}}_{z'} \oplus 0) \cong \tilde{\mathfrak{J}}_d$ and $\tilde{\mathfrak{J}}'' := \mathbb{P}(0 \oplus \tilde{\mathcal{L}}_{z''}) \cong \tilde{\mathfrak{J}}_d$ and let $\mathbb{J}_d^o := \mathbb{J}_d \setminus \{\tilde{\mathfrak{J}}' \cup \tilde{\mathfrak{J}}''\}$. Then the restriction $\mathbb{L}_d|_{\mathbb{J}_d^o \times C_0}$ is locally free. Moreover,*

$$\mathbb{J}_d^o \longrightarrow \mathsf{Pic}^d(C_0), \quad p \longmapsto [\mathbb{L}|_{\{p\}\times C_0}]$$

is an isomorphism.

(iii) *$\mathbb{L}_d|_{\tilde{\mathfrak{J}}' \times C_0} \cong (\mathrm{id}_{\tilde{\mathfrak{J}}'}, n)_*(\mathcal{L}(-z'))$, and $\mathbb{L}_d|_{\tilde{\mathfrak{J}}'' \times C_0} \cong (\mathrm{id}_{\tilde{\mathfrak{J}}''}, n)_*(\mathcal{L}(-z''))$.*

Proof. The first statement is clear since $\mathcal{L}$ is trivialised along $\tilde{\mathfrak{J}}_d \times \{\tilde{\zeta}_0\}$. Since the normalization morphism is isomorphism away from z' and z'', we deduce that $\mathbb{L}_d$ is locally free away from $\mathbb{J}_d \times \{z\}$. Let us choose, locally over $\tilde{\mathfrak{J}}_d$, an isomorphism $\sigma : \tilde{\mathcal{L}}_{z'} \to \tilde{\mathcal{L}}_{z''}$. Using this identification, we find that locally over $\tilde{\mathfrak{J}}_d$, at the point $[1 : \varepsilon] \times \{z\} \in \mathbb{P}^1 \times C_0$, we have the situation

$$0 \longrightarrow (\mathbb{L}_d)_\varepsilon \longrightarrow n_*\tilde{\mathcal{L}} \xrightarrow{h_\varepsilon = \mathrm{ev}_{z''} - \varepsilon(\sigma \cdot \mathrm{ev}_{z'})} \tilde{\mathcal{L}}_{z''} \longrightarrow 0. \tag{B.2}$$

Algebraically it looks as follows: consider the ring $R := \mathbb{C}[u, v]/(uv)$ with maximal ideal $\mathfrak{m} = (u, v)/(uv)$. We are interested in the exact sequence of $R_\mathfrak{m}$-modules

$$0 \longrightarrow (\mathbb{L}_d)_\varepsilon \longrightarrow (\mathbb{C}[u] \oplus \mathbb{C}[v])_\mathfrak{m} \xrightarrow{\psi_\varepsilon} \mathbb{C} \longrightarrow 0,$$

$$\frac{f(u)}{h(u,v)} \overset{h_\varepsilon}{\longmapsto} -\varepsilon\frac{f(0)}{h(0,0)}, \qquad \frac{g(v)}{h(u,v)} \overset{\psi_\varepsilon}{\longmapsto} \frac{g(0)}{h(0,0)}.$$

We obtain that

$$(\mathbb{L}_d)_\varepsilon = \left\{ (r_1, r_2) \,\middle|\, r_1 = \frac{a + uf(u)}{1 + h(u,v)},\ r_2 = \frac{\varepsilon a + vg(v)}{1 + h(u,v)},\ a \in \mathbb{C},\ h \in \mathfrak{m} \right\},$$

and one checks immediately that $(\mathbb{L}_d)_\varepsilon \cong R_\mathfrak{m}$ as modules, for $\varepsilon \neq 0, \infty \in \mathbb{P}^1$.

The third statement follows from this very computation, as it corresponds to $\varepsilon = 0$ and ∞ respectively, but it can be seen immediately from the exact sequence (B.2) too. □

Obviously, for any line bundle $L \to \tilde{C}_0$, $L(-z') = \mathcal{O}(z'' - z') \otimes L(-z'')$; consequently, in order to obtain the variety which parameterizes the isomorphism classes of rank one, torsion free sheaves on C_0, we must identify $\tilde{\mathfrak{J}}', \tilde{\mathfrak{J}}'' \hookrightarrow \mathbb{J}_d$ by the map $[L] \longmapsto [L(z'' - z')]$. We denote by $\mathfrak{J}_d$ the resulting variety.

Theorem B.1 (Igusa). *The compactified Jacobian $\mathfrak{J}_d$ of C_0 parameterizes equivalence classes of torsion free, rank one sheaves over C_0, of degree d, and the sheaf $\mathbb{L}_d$ constructed in the previous proposition descends to a universal Poincaré sheaf $\mathfrak{L}_d \to \mathfrak{J}_d \times C_0$, trivialised along $\mathfrak{J}_d \times \{\zeta_0\}$.*

In particular, we recover Igusa's result in [16, page 187] that $\mathbb{J}_d \to \mathfrak{J}_d$ is the normalisation map. Let us point out that there is a rational map $\mathfrak{J}_d \dashrightarrow \tilde{\mathfrak{J}}_d$, which fits into the diagram

$$\begin{tikzcd} & \mathbb{J}_d \arrow[dl] \arrow[dr] & \\ \mathfrak{J}_d \arrow[rr, dashed] & & \tilde{\mathfrak{J}}_d, \end{tikzcd} \tag{B.3}$$

and which is well defined on $\mathsf{Pic}^d(C_0) \subset \mathfrak{J}_d$.

We conclude this part by recalling the result concerning the existence of the *relative* compactified Jacobian of a flat family of reduced and irreducible curves. This is a difficult topic, but fortunately it was solved in [1, theorem 3.4] in a much wider generality.

Theorem B.2. *Let us consider a flat family of reduced curves $\mathscr{C} \to \triangle$ whose arithmetic genus is g, such that for all points $t \in \triangle \setminus \{o\}$ the fibre C_t is smooth and irreducible, and let d be an integer.*

Assume that:

(a) *the central fibre C_o is irreducible and has exactly one node;*
(b) *there is a section $\sigma : \triangle \to \mathscr{C}$ such that the point $\sigma(o) \in C_o$ is smooth.*

Then there exists the relative compactified Jacobian $\mathscr{J} \to \triangle$, and a universal sheaf $\mathscr{L} \to \mathscr{J} \times_\triangle \mathscr{C}$ of relative degree d having the properties:

(1) *the fibre $\mathscr{J}_o$ of $\mathscr{J} \to \triangle$ over $o \in \triangle$ is the compactified Jacobian constructed in theorem B.1, and the restriction $\mathscr{L}_o \to \mathscr{J}_o \times C_o$ coincides with the corresponding Poincaré sheaf;*
(2) *$\mathscr{L}$ is trivialised along $\mathscr{J} \times_\triangle \sigma(\triangle)$.*

Part II: *degeneration to a curve with two components* Let C_0 be a connected curve with two smooth and irreducible components C' and C'' which meet at one ordinary double point $Q \in C_0$. The normalisation of C_0 consists of the disjoint union $C' \sqcup C''$, and we denote by $Q' \in C'$ and $Q'' \in C''$ the points above Q. Line bundles over C_0 are topologically characterized by the bi-degree (d', d'') of their restriction to C' and C'' respectively. We assume that $d' > 2g(C') - 2$ and

$d'' > 2g(C'') - 2$ and consider $\mathcal{L}' \to \mathfrak{J}'_{d'} \times C'$ and $\mathcal{L}'' \to \mathfrak{J}''_{d''} \times C''$ the Poincaré bundles of degree d' and d'' trivialised at Q' and Q'' respectively. Then the push-forwards

$$\begin{array}{ccc} \mathcal{L}' & & n'_*\mathcal{L}' \\ \downarrow & & \downarrow \\ \mathfrak{J}'_{d'} \times C' & \xrightarrow{n'} & \mathfrak{J}'_{d'} \times C_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}'' & & n''_*\mathcal{L}'' \\ \downarrow & & \downarrow \\ \mathfrak{J}''_{d''} \times C'' & \xrightarrow{n''} & \mathfrak{J}''_{d''} \times C_0, \end{array}$$

are locally free, and we define $\mathcal{L} \to \left(\mathfrak{J}'_{d'} \times \mathfrak{J}''_{d''}\right) \times C_0$ by means of

$$0 \longrightarrow \mathcal{L} \longrightarrow n'_*\mathcal{L}' \oplus n''_*\mathcal{L}'' \xrightarrow{\mathrm{ev}_{Q'} - \mathrm{ev}_{Q''}} \mathbb{C}_Q \longrightarrow 0.$$

Theorem B.3. (i) *The homomorphism*

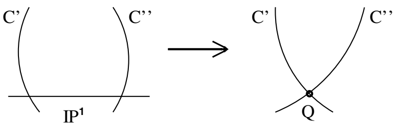
$$\mathfrak{J}_{(d',d'')} := \mathsf{Pic}^{(d',d'')}(C_0) \xrightarrow{n^*} \mathsf{Pic}^{d'}(C') \times \mathsf{Pic}^{d''}(C'')$$

is an isomorphism; in particular, $\mathsf{Pic}^{(d',d'')}(C_0)$ *is projective.*

(ii) $\mathcal{L} \to \mathfrak{J}_{(d',d'')} \times C_0$ *is locally free, and is the Poincaré bundle trivialised at the double point* Q.

Proof. See for instance [21, proposition 10.2]. □

This description is not satisfactory since $\mathcal{L}$ is trivialised precisely at the singular point of C_0, and this is inconvenient when dealing with families of curves. So, instead of C_0 we will prefer the curve $C' + C''$ obtained by inserting a $\mathbb{P}^1$ at Q; this $\mathbb{P}^1$ will play the role of the 'ghost bubble' in the Gromov-Witten theories. The curve $C' + C''$ has three irreducible components, and there is a canonical morphism $C' + C'' \to C_0$ which contracts the bubble component.



The 'nice Poincaré sheaf' is the pull-back $\hat{\mathcal{L}}$ of $\mathcal{L}$ by the morphism $\mathfrak{J}_{(d',d'')} \times (C' + C'') \to \mathfrak{J}_{(d',d'')} \times C_0$; clearly, its restriction to the bubble component is trivial.

Our final goal is to recall the result about the existence of the *relative* compactified Jacobian for a flat family of irreducible curves degenerating to a curve with two components. The difficulty in this case relies in the fact that the corresponding Picard functor is not separated, and therefore can not be representable. This difficulty was fortunately solved by Esteves in [11].

Theorem B.4. *We consider the integers* d', d'' *and* $g', g'' \geq 1$, *and let* $d = d' + d''$, $g = g' + g''$. *Let us consider a flat family of reduced curves* $\mathscr{C} \to \triangle$ *whose arithmetic genus is* g, *such that for all points* $t \in \triangle \setminus \{o\}$ *the fibre* C_t *is smooth and irreducible.*

Assume that:

(a) *The central fibre is reducible and has three smooth, irreducible components C', C'' and $\mathbb{P}^1$ of genus g', g'' and* 0 *respectively, which meet at ordinary double points, such that $C' \cap C'' = \emptyset$;*

(b) *There are sections $s_0, s', s'' : \triangle \to \mathscr{C}$ having the property that $s'(o) \in C'$, $s''(o) \in C''$ and $s_0(o) \in \mathbb{P}^1$ are smooth points.*

Then there exists a relative compactified Jacobian $\mathscr{J} \to \triangle$ and a universal sheaf $\mathscr{L} \to \mathscr{J} \times_\triangle \mathscr{C}$ having the properties:

(1) *the fibre $\mathfrak{J}_o$ of $\mathscr{J} \to \triangle$ over o is the Jacobian $\mathfrak{J}_{d',d''}$ constructed in theorem B.3* (i)*, and the restriction to $J_{d',d''} \times (C' + C'')$ is the line bundle $\hat{\mathfrak{L}}_0$ constructed in the remark following theorem B.3* (ii)*;*

(2) *$\mathscr{L}$ is trivialised along $\mathscr{J} \times_\triangle s_0(\triangle)$.*

Proof. Following [11], we define first an appropriate vector bundle $\mathcal{E} \to \mathscr{C}$, called *a polarisation*, and we will consider only the line bundles which are (semi-)stable with respect to this one. The (semi-)stability with respect $\mathcal{E}$ forces line bundles of degree d on the smooth curves $(C_t)_{t\in\triangle}$ to degenerate into line bundles having 3-degree $(d', 0, d'')$ on the special fibre $\hat{C}_o$ as $t \to o$.

We may assume without restricting the generality that $d' \geq 2g' - 1, d'' \geq 2g'' - 1$, otherwise we twist the whole situation by $\mathcal{O}_{\mathscr{C}}(n_1 s'(\triangle) + n_2 s''(\triangle))$, with $n_1, n_2 \gg 0$. In this situation, we let $A' := 3(d' - g') + 1 > 0$, $A'' := 3(d'' - g'') + 1 > 0$ and $A_0 := 1$, and consider, according to [11, observation 57], the polarisation

$$\mathcal{E} := \big[\mathcal{O}_{\mathscr{C}} \oplus \mathcal{O}_{\mathscr{C}}\big(-A's'(\triangle) - A''s''(\triangle) - A_0 s_0(\triangle)\big)\big]^{\oplus t}, \quad t \gg 0.$$

On smooth fibres, the $\mathcal{E}$-(semi-)stability property for line bundles is vacuous, but controls *their degeneration* to the central fibre. It is immediate to see that for this polarisation, the $\mathcal{E}$-semi-stability and $\mathcal{E}$-stability properties on $C' + C''$ coincide, and therefore the relative compactified Jacobian $\mathscr{J} \to \triangle$ consisting of degree d, simple, torsion-free, rank one sheaves on $\mathscr{C}/\triangle$, which are $\mathcal{E}$-stable is separated and proper over $\triangle$ (*cf.* [11, theorem 32]). The central fibre $\mathfrak{J}_o$ of $\mathscr{J} \to \triangle$ is projective and irreducible (see [11, example 41, (1)] for $\delta = 1$), and it contains $\mathfrak{J}_{d',d''}$ constructed in B.3. Consequently they are isomorphic.

The existence of the universal sheaf follows from the fact that the relative compactified Jacobians constructed by Esteves are fine moduli spaces (in the étale topology). □

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